PHYS 460 Project 2: Conservative Chaos in Classical 1-D Elastic Scattering

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We report on the numerical study of conservative chaotic motion in a classical scattering example in which the state of two point particles with independent masses are calculated in a closed system. It is found that for certain initial conditions the resulting elastic collisions with each other and the ground can lead to chaos. To show the chaotic nature of the system Poincaré cross sections are taken at the moment of collision between the two masses. Combined with basic plots for the positions of the particles it is seen that different values of the masses produce different "degrees" of chaos. In addition, the autocorrelation function is employed, showing the diminishing similarity present for chaotic systems. All calculations are done by exactly solving for the moments of collision and then filling in numerically calculated values for the positions.

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I. INTRODUCTION

In the late 19th century Henri Poincaré's work on the infamous 3-body problem gave birth to modern chaos theory, and since then numerous chaotic systems have been discovered. In order to evaluate these systems we must understand how to analyze chaos, and what we can learn from it. Chaos, not to be confused with random, describes a system in which a very small perturbation in the initial conditions can lead to large changes in the system in a short period of evolution. Since initial conditions can only be measured with a finite accuracy this implies that the system will be almost totally unpredictable within a short period of time. In order for chaos to arise there are two conditions: the system must be nonlinear, and must have 3 or more degrees of freedom as defined by dynamical system theory.

In our study of chaos we simulate a simple scattering scenario between two point masses. The system is composed of two masses released with with some initial conditions specified by $x_{10}, x_{20}, v_{10}, v_{20}$ which are then allowed to free fall under the influence of gravity. The only constraint on the system is a "floor" placed at $x = 0$ which allows for repeated collisions between the two particles. In between collisions the particles are governed by the simple solution to Newton's 2nd Law, $F = ma$, for a constant force f. Given that $f = -g$ we have the following equation of motion for particle n,

$$X_n(t) = -\frac{g}{2}t^2 + v_{n0}t + x_{n0}. \tag{1}$$

All collisions are considered to be elastic, making the system conservative, and are governed by the equations for conservation of kinetic energy (for elastic collisions) and conservation of momentum, Eqs. (2) and (3) respectively:

$$\begin{aligned} \frac{m_1 v_1}{2} + \frac{m_2 v_2}{2} &= \frac{m_1 u_1}{2} + \frac{m_2 u_2}{2}, & (2)\\ m_1 v_1 + m_2 v_2 &= m_1 u_1 + m_2 u_2, & (3) \end{aligned}$$

where v_n is the velocity of the particle before the collision, and u_n is the velocity afterwards. Simple algebraic manipulation of Eqs. (2) and (3) gives the following expressions for u_1 and u_2 after the two masses collide,

$$\begin{aligned} u_1 &= \frac{v_1(m_1 - m_2) + 2m_2 v_2}{m_1 + m_2}, & (4)\\ u_2 &= \frac{v_2(m_2 - m_1) + 2m_1 v_1}{m_1 + m_2}. & (5) \end{aligned}$$

In the case of a collision with the floor we simply reverse the direction of the velocity, leaving its magnitude unchanged.

In order for this system to be chaotic it must meet the two criteria listed earlier, nonlinearity and a certain degree of freedom. The nonlinearity in this system is not particularly obvious from the equations given, but we must consider the fact that the entire state of the system is not given by Eq. (1), but rather by a piecewise continuous combination of them. In other words, the collisions create impulses in the velocity of the particles which could be modeled by nonlinear steps in $v(t)$. After each of these impulses the system again obeys Eq. (1) but with a different x_0 and v_0, and with time "starting" back at zero. The other requirement for chaos is at least 3 degrees of freedom as per dynamical system theory, in our case we have 4 dimensional freedom given by the equations for $X_1(t), X_2(t), V_1(t)$, and $V_2(t)$. Dynamical system theory requires first order differential equations, and we have:

$$\begin{aligned} \frac{dX_1}{dt} &= V_1, & (6)\\ \frac{dX_2}{dt} &= V_2 & (7)\\ \frac{dV_1}{dt} &= -g, & (8)\\ \frac{dV_2}{dt} &= -g. & (9) \end{aligned}$$

Given that these requirements are met our system is capable of chaos. This does not mean it is always chaotic,

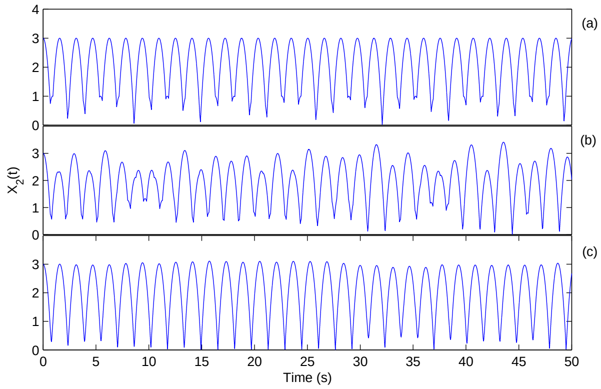

FIG. 1: Trajectories of mass m_2 for (a)$m_2/m_1 = 1$, (b)$m_2/m_1 = 2$, and (c)$m_2/m_1 = 9$.

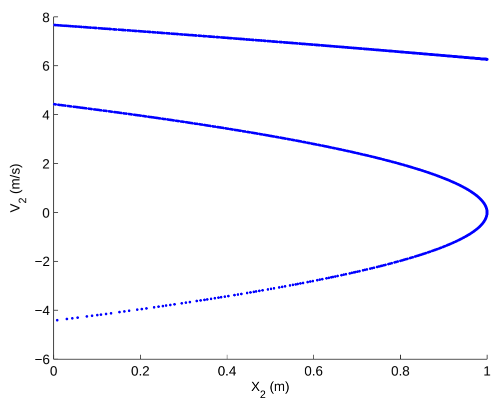

FIG. 2: Poincaré cross-section for $m_2/m_1 = 1$.

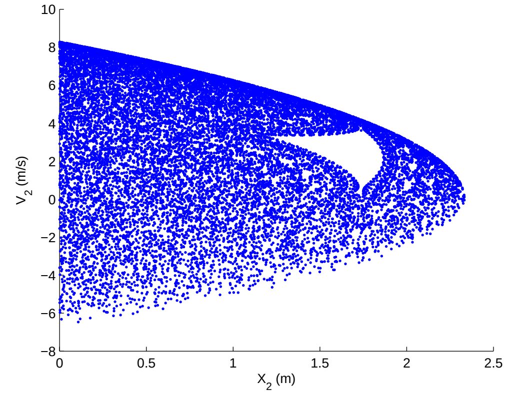

FIG. 3: Poincaré cross-section for $m_2/m_1 = 2$.

as we shall see, but that given the proper conditions chaos can arise. It is also important to note that because this system is conservative the chaos our system exhibits will be different than that of a dissipative system. These two types of chaos are simply called conservative and dissipative chaos respectively. After describing the method we use to evaluate the system we will begin discussion and analysis of the most fundamental tool in chaos theory, the Poincaré cross section. We will also examine the aperiodic structure of chaos via an implementation of the Autocorrelation function in order to visualize the signal's self-similarity.

II. METHOD

The computational implementation of the system is fairly simple. The next moments of collision for the particles, both with each other and the floor, are computed exactly for the given state. From these collisions we choose the closest collision, i.e. the one that will happen first. After obtaining the point at which the current iteration of Eq. (1) becomes invalid we then compute all values of X_1, X_2, V_1, and V_2 up until that point in a specified discrete time step. After this is complete the next state of the system is then calculated via Eqs. (4) and (5) for mass-mass collisions, or the simple reversal of velocity mentioned in Sec. I if the collision is with the floor. The entire state of the system is also stored if the collision is between the two masses since this data is needed for the Poincaré cross-sections mentioned in Sec. I.

The moments of collision are calculated by both simultaneously solving Eq. (1) for the particles to find their moment of collision, and by setting each equal to zero to solve for the floor impact time. The solutions of the quadratic equation are done in such a way as to avoid dangerous computational errors and with special cases for $x_0 = 0$ and $v_0 = 0$. For all data presented the system was evolved for $t = 100$ s with a discrete time step of $\Delta t = 10^{-4}$ s.

III. CHAOS

We now begin our analysis of the system by evolving for several different sets of initial conditions. For all simulations (except one) the the initial conditions of $x_{10} = 1, x_{20} = 3, v_{10} = 0, v_{20} = 0$ are shared, we will simply vary the mass ratio m_2/m_1 to study the system. Specifically we will take m_2/m_1 to be 1,2, and 9 in our three different simulations.

We start by plotting the trajectory of the second, heavier, particle versus time which can be seen for all three masses in Fig. 1. Here we see a structure being very close to periodic for $m_2/m_1 = 1$ and $m_2/m_1 = 9$. For $m_2/m_1 = 2$, on the other hand, the system seems very aperiodic. These initial plots would lead us to believe that $m_2/m_1 = 2$ may be exhibiting "chaotic behavior", and possibly $m_2/m_1 = 9$ as well, looking at the variations at the maximum of each crest. Overall, the system is obviously chaotic from these plots, a slight increase in the mass changed the system from almost periodic to completely aperiodic, and then back again. What is

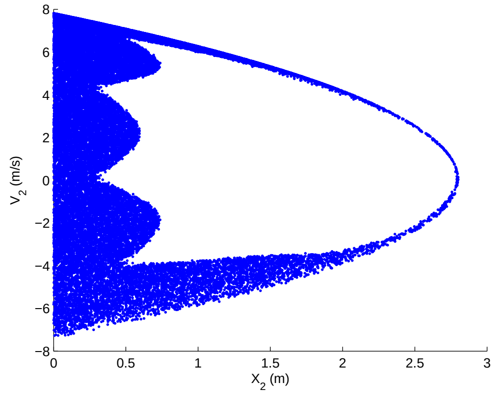

FIG. 4: Poincaré cross-section for $m_2/m_1 = 9$.

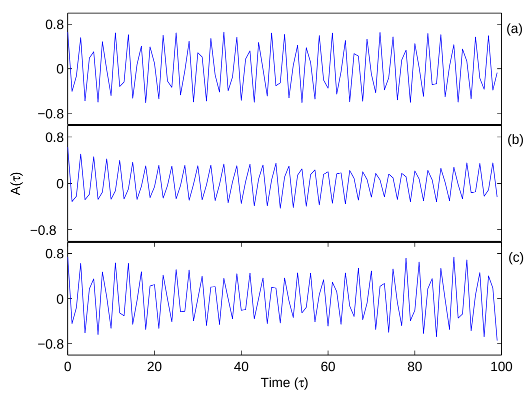

FIG. 5: Autocorrelation function of mass m_2 for (a)$m_2/m_1 = 1$, (b)$m_2/m_1 = 2$, and (c)$m_2/m_1 = 9$.

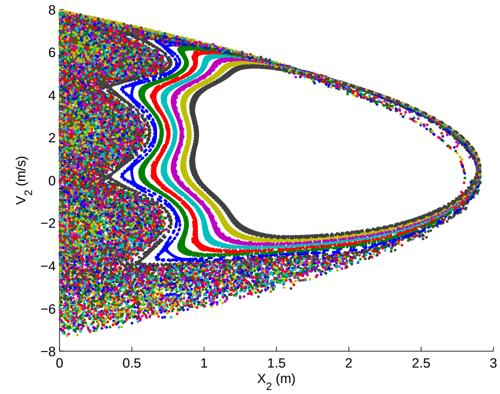

FIG. 6: Poincaré cross-section for $m_2/m_1 = 9$ for multiple initial conditions.

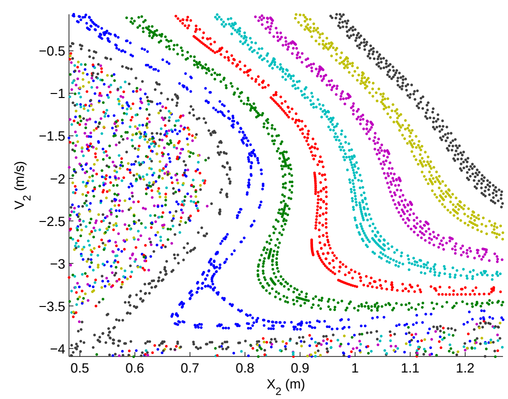

FIG. 7: Enlarged view of Fig. 6.

meant by chaotic behavior is that the particular evolution of the system is displaying what at first sight looks like unpredictability, the peaks appear to be random.

We now seek to prove that the system is chaotic, and not random. In order to do this we have computed the Poincaré cross-section of our system at the moment of collision for X_2. These sections can be seen in Figs. 2, 3, and 4. These so-called Poincaré cross-sections give us a glimpse into the world of chaos, they represent a slice through the phase space of the system, which for conservative chaos, is a "fat" fractal structure. Here we clearly see that our system is not random, the apparent order in the phase space would not be there for a random system.

This "fat" fractal structure can be seen more clearly in Figs. 6 and 7 which show the Poincaré section for many different initial conditions for the mass ratio $m_2/m_1 = 9$. Here we can see that these structures, are very ordered, not just repeating loops but very intricate patterns that comprise the fractal structure

These the figures (Figs. 2, 3, and 4) show that although there are many combinations of x and v, not all are valid, there are regions seen in the phase space that the chaotic system does not allow. These "regular islands" are regions of stability for the system, these points are not part of the chaotic region. That is to say, for the given initial conditions the system will never take on values in that region. The only way for the particles to evolve inside these "islands" are for regular, not chaotic, motion. The regions seen in these plots exaggerate the regular islands because we are only computing a small part of the fractal structure, the chaotic band may extend into this space for a different set of initial velocities or positions. However, for the computed conditions, these islands do depict areas the system will never occupy.

Specifically we see in Figs. 3 and 4 that the phase space is very densely packed with points, where as Fig. 2 is much simpler looking. This further proves our conjecture that $m_2/m_1 = 2$ and $m_2/m_1 = 9$ were chaotic. The ratio $m_2/m_1 = 1$ (Fig. 2), on the other hand, seems to show some form of period doubling, a sign of the onset of chaos. In this state there only a few values of v for each value of x, instead of the many seen in the other two cases. As stated earlier, stating that a system is chaotic does not mean it always exhibits chaotic behavior, simply that it is capable of it.

IV. AUTOCORRELATION FUNCTION

As we stated in Sec. I chaotic systems are very aperiodic and the signal should not show any significant similarity with itself as time progresses. In order to quantitatively evaluate this for our system we employed the Autocorrelation function, which for continuous signals is given by the integral,

$$C(\tau) = \int_0^\infty (x(t) - \bar{x})(x(t+\tau) - \bar{x})dt. \tag{10}$$

Converting this to a discrete, programmable form, we have,

$$A[r] = \sum_{i=1}^{N-r\Delta t} (x[i] - \bar{x})(x[i + r\Delta t] - \bar{x}), \tag{11}$$

where N is the number of elements in the data set and Δt is the rate at which we are "sampling" the data to be compared, $r\Delta t$ is the τ from Eq. (10).

The results of evaluating this discrete sum for our different mass ratios is shown in Fig. 5. It is slightly difficult to see differences among the plots, but the correlation for $m_2/m_1 = 2$ in Fig. 5b is clearly smaller than that for $m_2/m_1 = 1$ and $m_2/m_1 = 9$ (Figs. 5a and 5c respectively). We also see that $m_2/m_1 = 1$ seems to be the most correlated, or most periodic, which agrees with our Poincaré cross-sections. Also, when inspecting Fig. 1b we see that $m_2/m_1 = 1$ appears to be the system most closely resembling a periodic structure. These results show that chaotic systems are aperiodic and have very little self-similarity, they are very unpredictable as time goes on, as seen by the slight decay in Fig. 5b.

V. CONCLUSIONS

By analyzing a simple chaotic system we are able to understand how chaotic systems evolve. We see that there is almost no predictability after a certain period of time for the system and that the systems are very aperiodic. Despite their apparent randomness in the spatial domain, there exists a very structured order in the phase space of these systems. Specifically their phase spaces take the form of fat fractals, displaying highly ordered, repetitive, structures. By analyzing these phase space diagrams we can better understand how the system may or may not evolve, and what properties it might have.

[1] Nicholas J. Giordano and Hisao Nakanishi, *Computational Physics* (Pearson Prentice Hall, Upper Saddle River NJ, 2006).