Interdisciplinary Topics in Complex Systems: Cellular Automata, Self-Organized Criticality and Neural Networks

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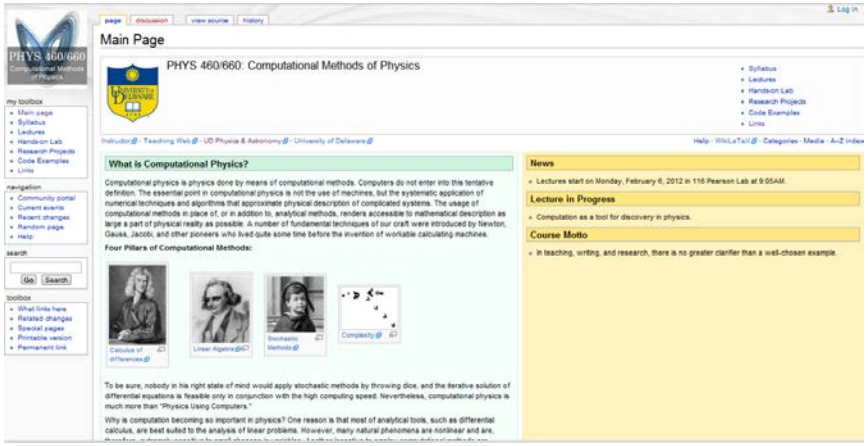
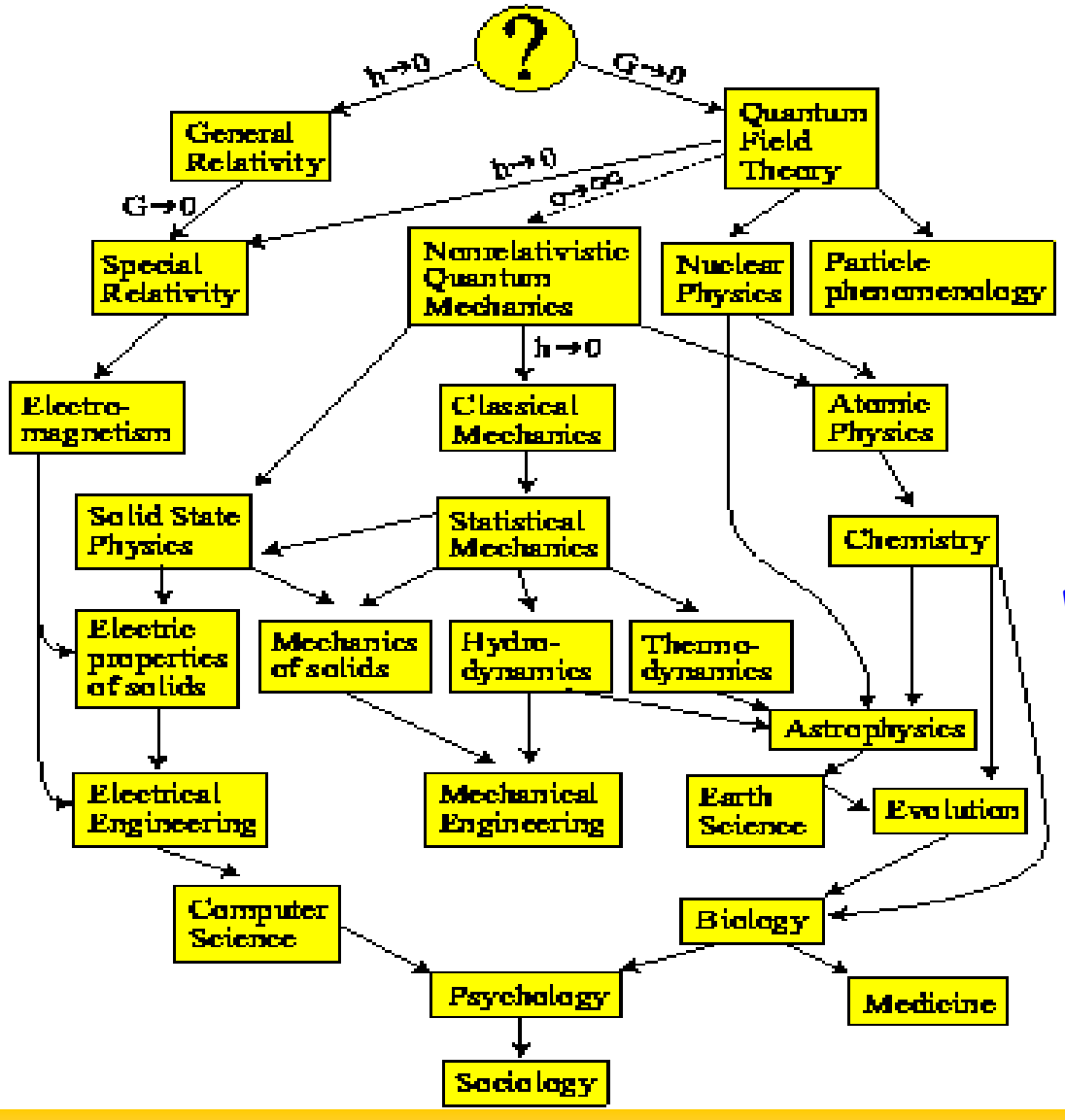
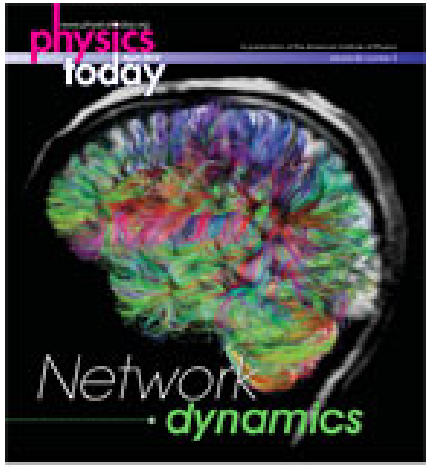
http://wiki.physics.udel.edu/phys660

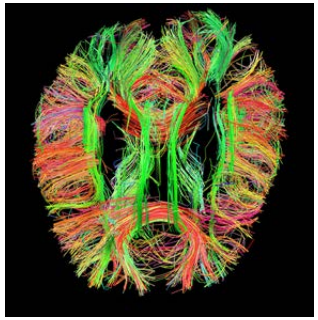


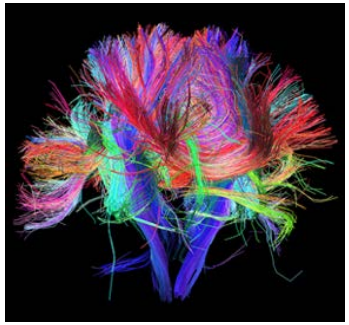
Complexity: From Quantum Gravity to Medicine and Sociology?





www.humanconnectomeproject.org





Can few simple laws of **Cellular Automata** produce rich behavior of complex systems?

- Cellular Automata: invented by von Neumann and Ulam in 1948 as an idealization of biological self-reproduction → this is why lattice site are called cells.
- More recently ***CAM*** have been applied to systems ranging from fluids to galaxies.
- ***CAM*** are example of a discrete dynamical system that can be simulated exactly on the computer.
- Discrete space, time, and physical quantities with integer values that are updated according to the local rules:

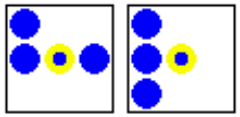
1. Space is discrete and there is a regular array of sites (cells). Each site has a finite set of values.
2. Time is discrete, and the value of each site is updated in a sequence of discrete time steps.
3. The rule for the new value of a site depends only on the values of a local neighborhood of sites near it.
4. The variables at each site are updated simultaneously ("synchronously") based on the values of the variables at the previous time step.

- CAM vs. discretized PDE: No accumulation of round-off errors.

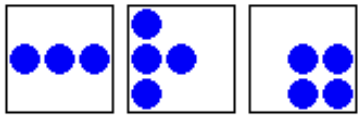
Perhaps The Most Famous CAM: Game of Life

- Invented by J. Conway in 1970 to produce fascinating patterns via few simple rules:

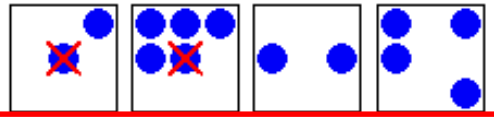
- A dead cell with exactly three live neighbors becomes a live cell (birth).

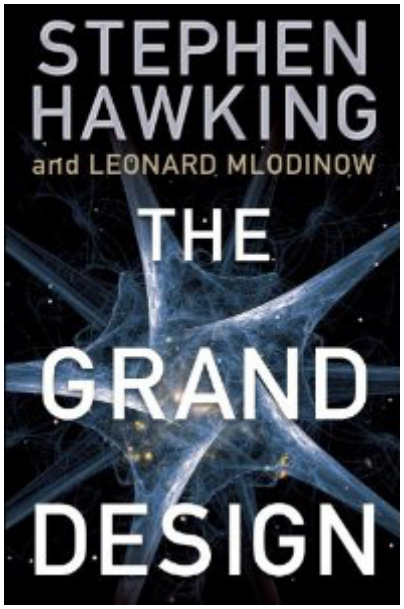


- A live cell with two or three live neighbors stays alive (survival).



- In all other cases, a cell dies or remains dead (overcrowding or loneliness).



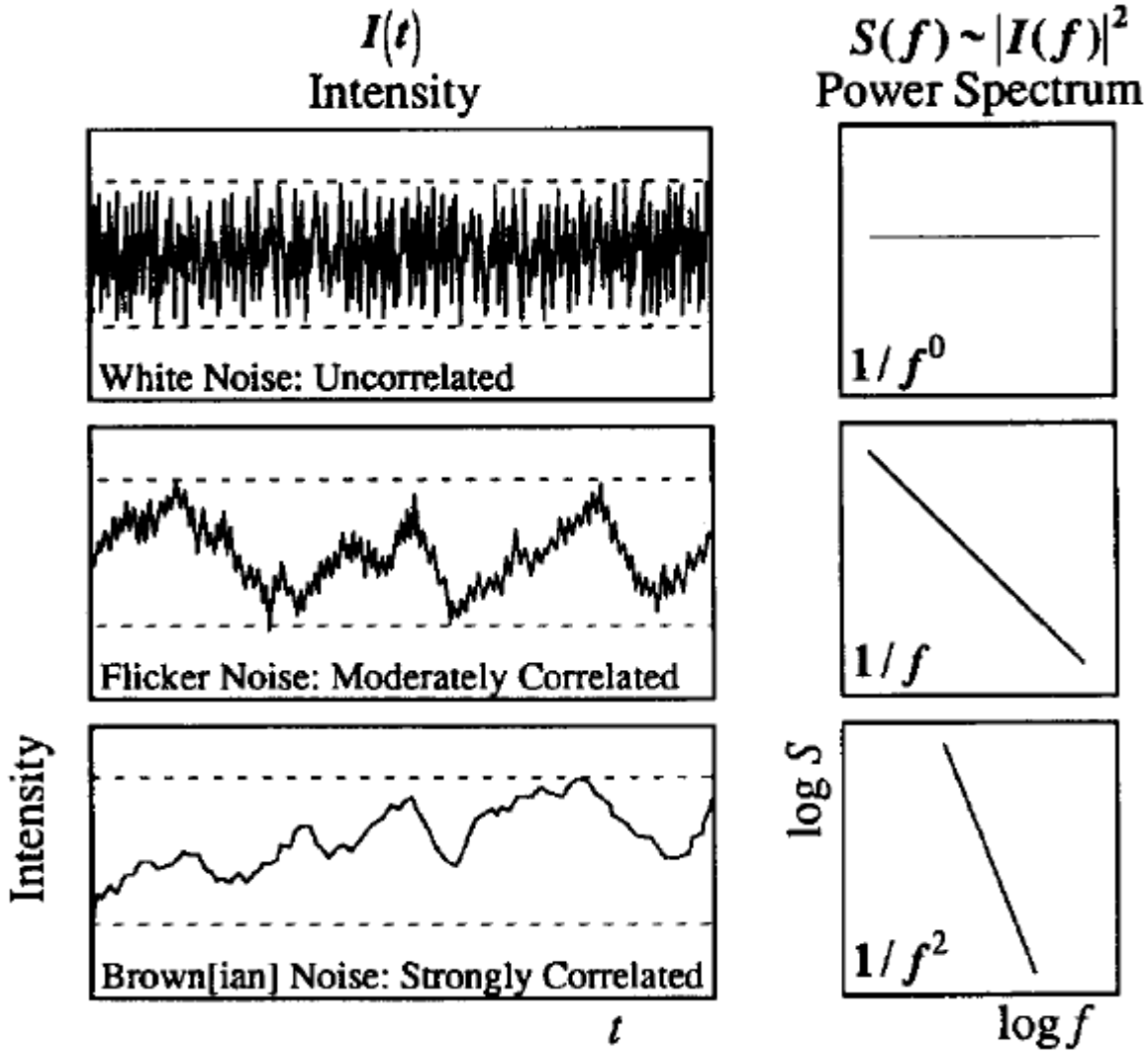


- Life is full of surprises → In most cases, it is impossible to look at a starting position and see what will happen in the future. The only way to find out is to follow the rules of the game.
- Life is one of the simplest examples of what is sometimes called "emergent complexity" or "self-organizing systems." It is the study of how elaborate patterns and behaviors can emerge from very simple rules.

Are There Universal Signatures of Complex Behavior?

- **Punctuated Equilibrium:** There are long periods of relative stasis punctuated by crises ("avalanched") of various sizes.
- **Power Laws:** The relationship between the sizes of these avalanches can be expressed in a simple exponential equation. There are no singular explanations for large events: the same forces that made the Dow Jones average drop five points yesterday also caused the crash of 1987.
- **Fractal Geometry:** Where a system exists in space, it is self-similar on all scales.
- **1/f Noise:** When a system evolves over a time, the record of evolution is also fractal.
- **Self-Organized Criticality:** explain all this phenomenological features by finding simple dynamics that spontaneously drives system into a **critical state characterized by power laws**.

Fluctuations of Physical Quantities: Noise and Its Power Spectrum



Power Laws and Scale Invariance

- **Main Idea:** In many complex systems large events are part of a distribution of events and do not depend on special conditions or external forces.
- If s represents the magnitude of an event, such as the energy released in an earthquake or the amount of snow in an avalanche, then a system is said to be critical if the number of events follows power law (for $\alpha \approx 1$ there is one large event of size 1000 for every 1000 events of size 1).

$$N(s) \sim s^{-\alpha}$$

- Power laws are scale invariant: $s \to bs \Rightarrow \tilde{N}(s) = \tilde{A}s^{-\alpha},\ \tilde{A} = Ab^{-\alpha}$
- Combining large number of independently acting random events gives Gaussian – **no scale invariance and practically no large events:**

$$N(s) \sim e^{-(s/s_0)^2} \Leftrightarrow s \to bs \Rightarrow \tilde{N}(s) \sim e^{-(s/s_1)^2},\ s_1 = s_0 b^2$$

Sandpile (Bak-Tang-Weisenfeld) Cellular Automaton

- A model of dissipative dynamical system with local interacting degrees of freedom and many metastable states

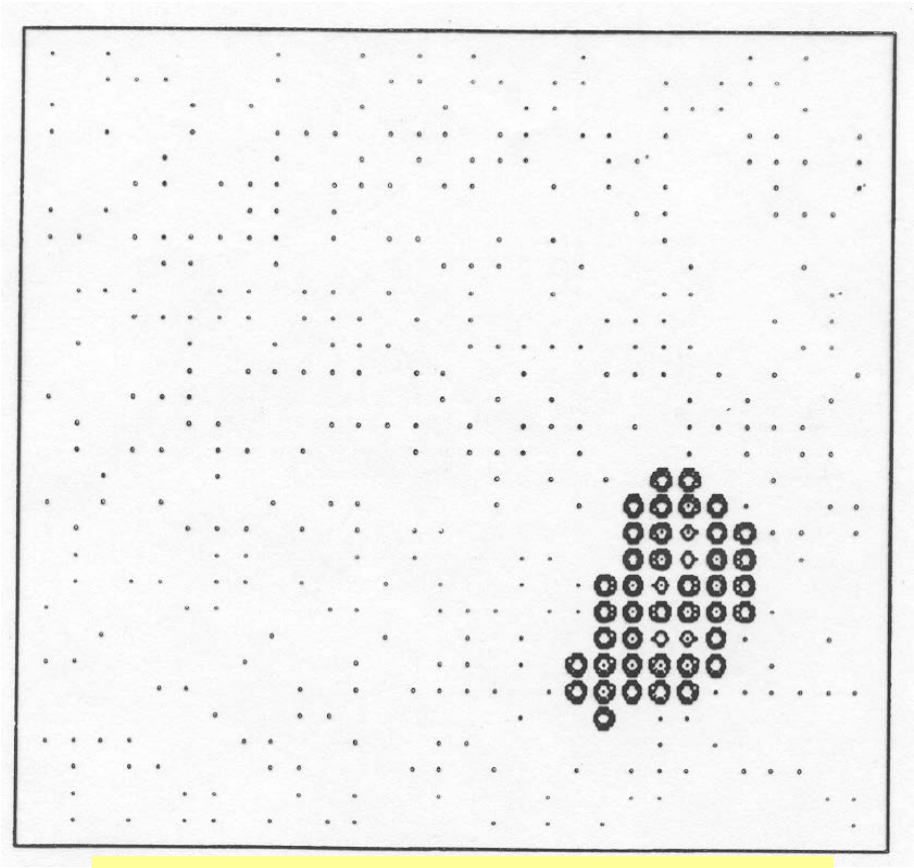
$$\text{If } z > 3 \Rightarrow \begin{cases} z(x,y) \to z(x,y) - 4 \\ z(x \pm 1, y) \to z(x \pm 1, y) + 1 \\ z(x, y \pm 1) \to z(x, y \pm 1) + 1 \end{cases}$$

- It self-organizes - without fine tuning of parameters, like temperature in the case of thermal critical phenomena, and independently of the initial conditions - into a critical state (scale-invariance) that is attractor of the dynamics and robust with respect to variations of parameters and the presence of quenched randomness.

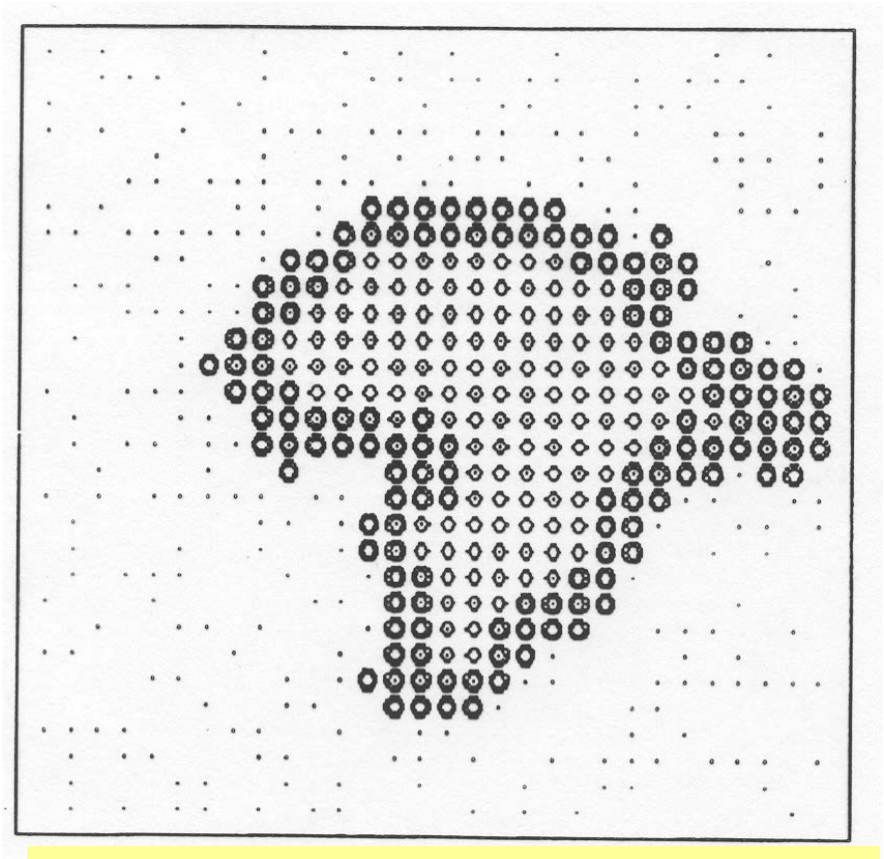
SOC	Ising Model
Continuous spontaneous flow	Magnetization M
Current of incoming particles	Magnetic Field H
$\langle z \rangle_c - \langle z \rangle$	Reduced Temperature $(T - T_c)/T_c$

- Constant average height in the critical state → the probability that activity will die is on the average balanced by the probability that activity will branch out.

Avalanches in the SOC state of Sandpile CAM



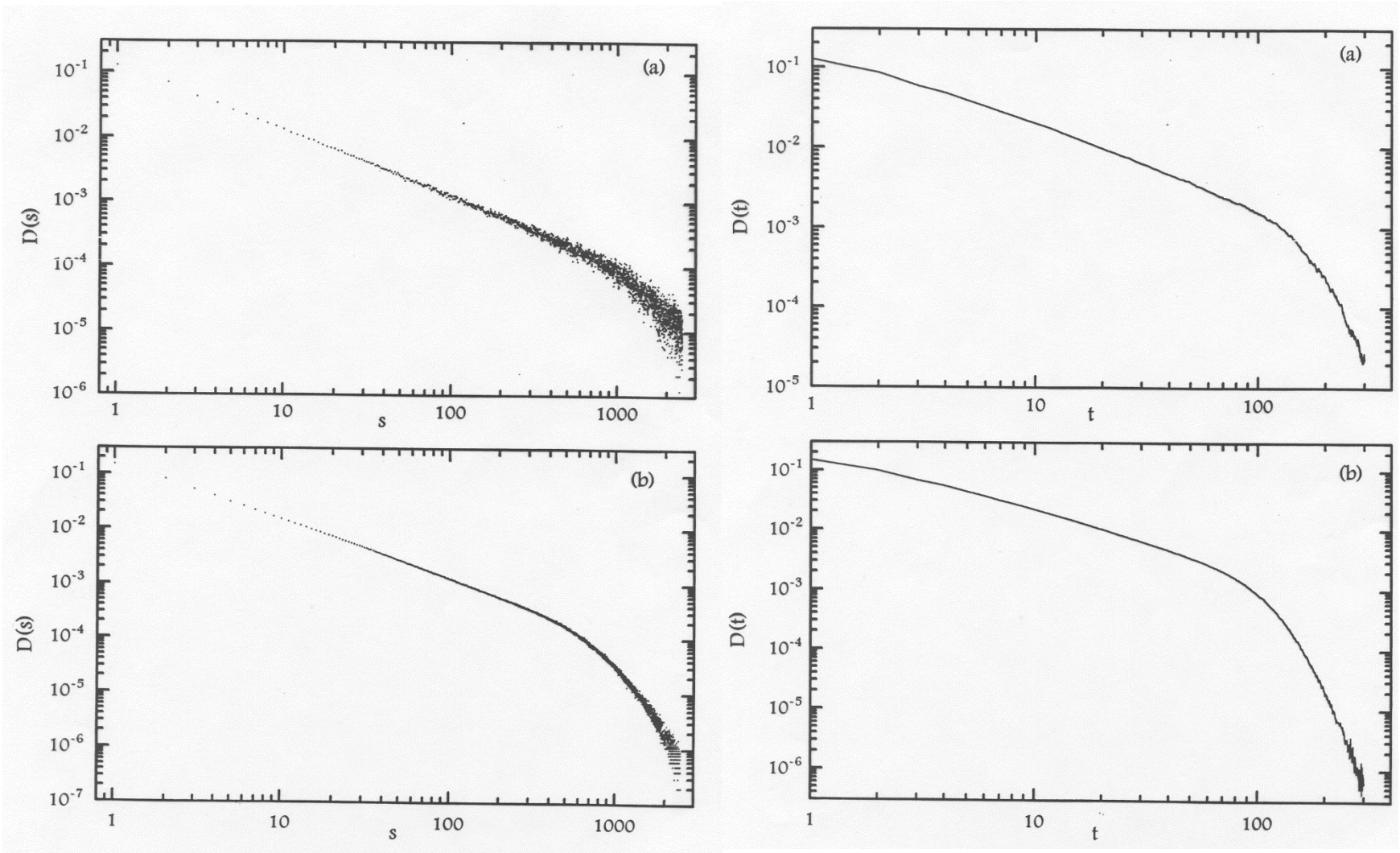
$$s = 25\ (a = 45)$$



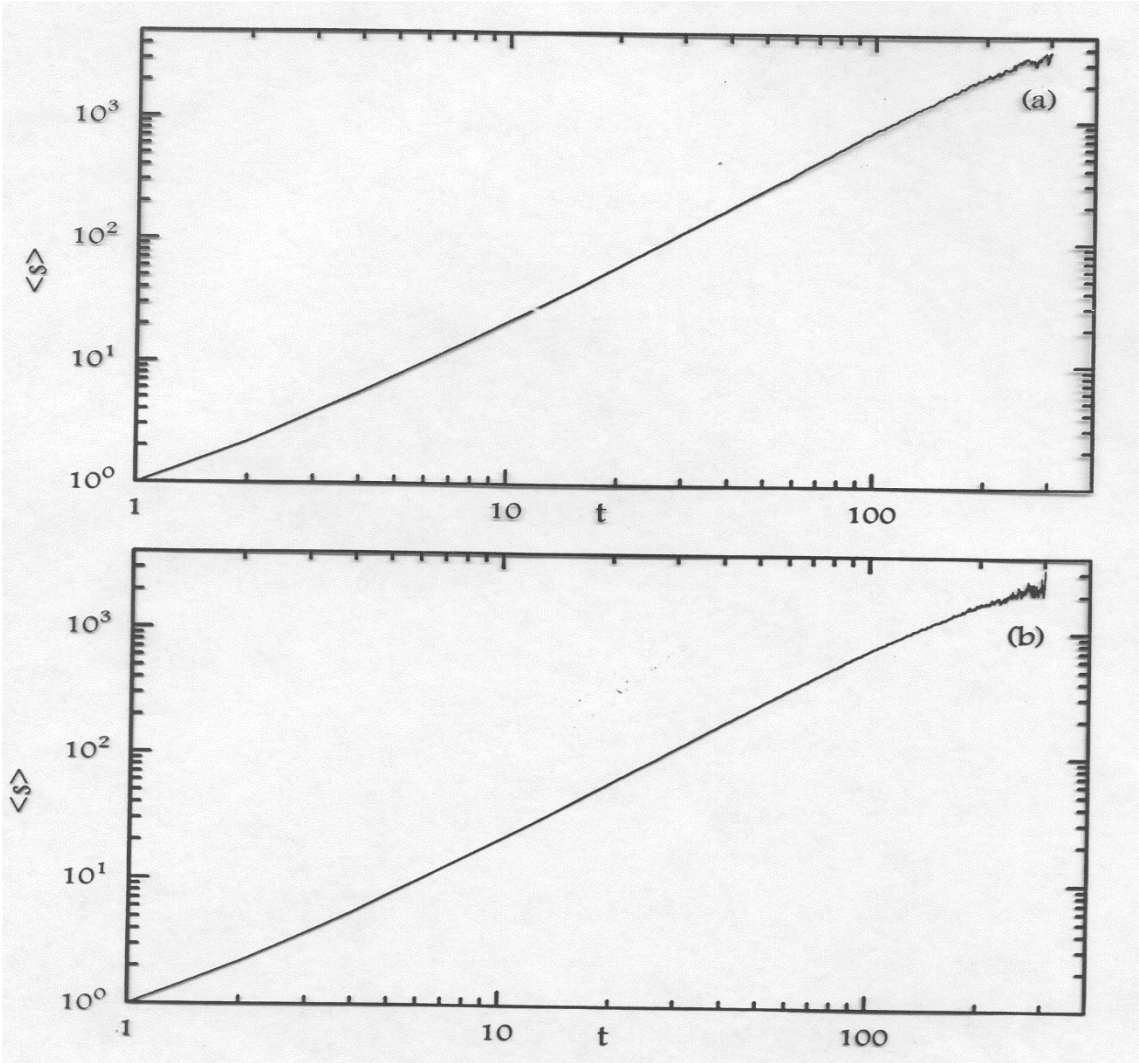
$$s = 197\ (a = 270)$$

Power Laws ("Scaling") in the SOC state of Sandpile CAM

Ensemble vs. Time Average



Size of Clusters vs. Time



How to Relax Sandpile to SOC state

```
%program soc_relax.m
%PARAMETERS
 N=128; %lattice size
%VARIABLES
  flag=0; % logical variable

  z(2:N+1,2:N+1)=randi([5,10],N,N);

  time=1;
  while 1 % infinite loop

    flag=0;
    z(1,:)=0; z(:,1)=0;
    z(N+2,:)=0; z(:,N+2)=0;
```

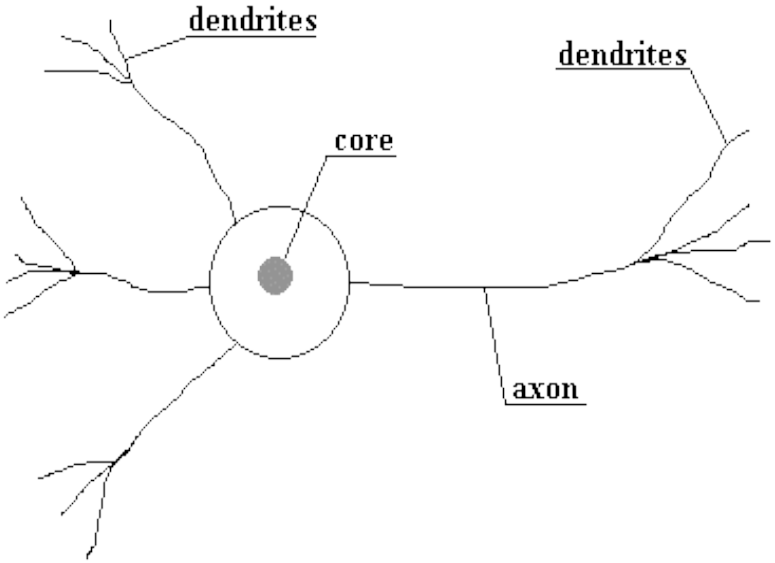
```
for i=2:N+1
   for j=2:N+1
      if (z(i,j)>3)
         z(i,j)=z(i,j)-4;
         z(i+1,j)=z(i+1,j)+1;
         z(i-1,j)=z(i-1,j)+1;
         z(i,j+1)=z(i,j+1)+1;
         z(i,j-1)=z(i,j-1)+1;
         flag=1;
      end
   end
end
   if (flag==0)
      break
   end
zaverage(time)=sum(sum(z(2:N+1,2:N+
1)))/(N*N);
time=time+1;
end % infinite loop
t=1:time-1;
plot(t,zaverage);
```

Neural Network vs. Human Brain

- Neural Net is an artificial representation of the human brain that tries to simulate its learning process.

- The term "artificial" means that neural nets are implemented in computer programs that are able to handle large number of necessary calculations during the learning process.

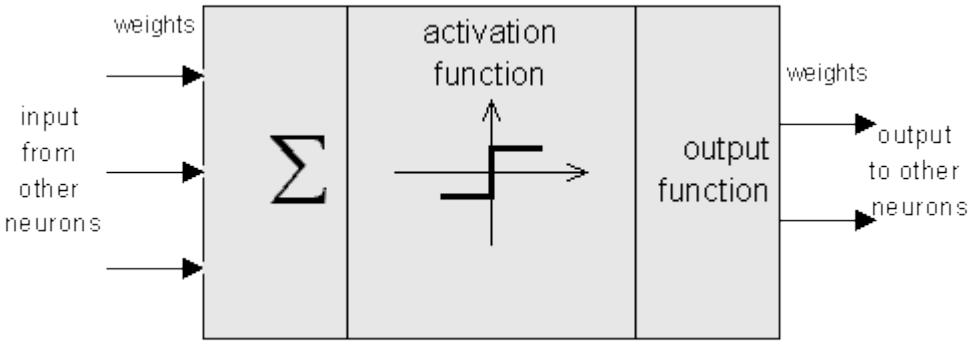
- Human brain consists of a large number (more than a billion) of neural cells that process information. Each cell works like a simple processor and only the massive interaction between all cells and their parallel processing makes the brain's abilities possible:



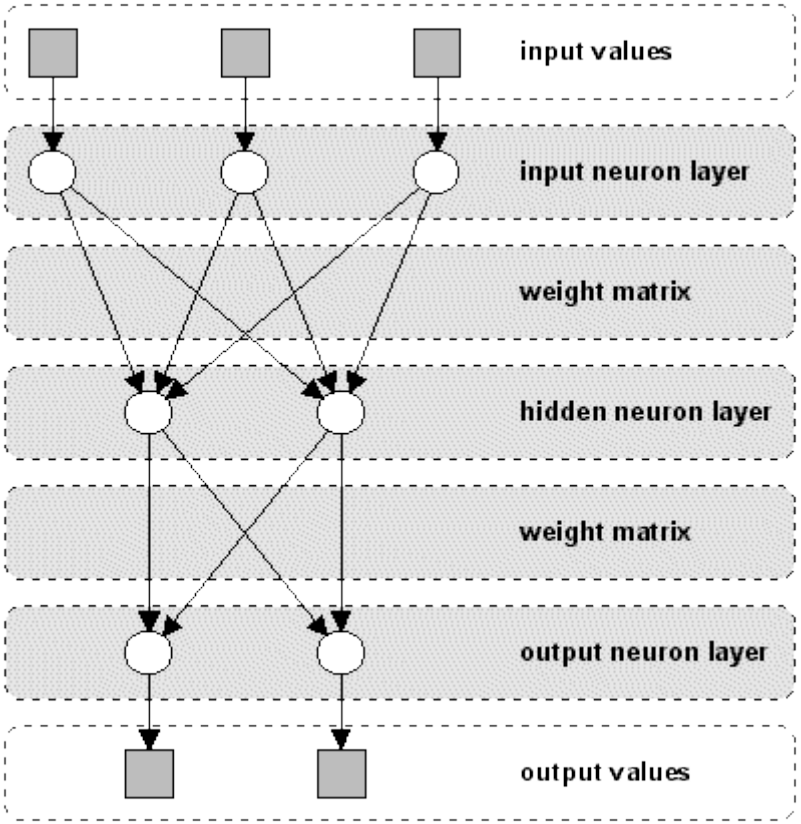
Neuron consists of a **core**, **dendrites** for incoming information and an **axon** with dendrites for outgoing information that is passed to connected neurons. Information is transported between neurons in form of electrical stimulations along the dendrites. Incoming information that reaches dendrites is added up and then delivered along axon to the dendrites at its end, where the information is passed to other neurons if the stimulation has exceeded a certain threshold. In this case, the neuron is said to be activated. If the incoming stimulation had been too low, the information will not be transported any further. In this case, the neuron is said to be inhibited. The connections between the neurons are adaptive, what means that the connection structure is changing dynamically. It is commonly acknowledged that the learning ability of the human brain is based on this adaptation.

Building Blocks of Neural Nets

Structure of a neuron of a Neural Net:



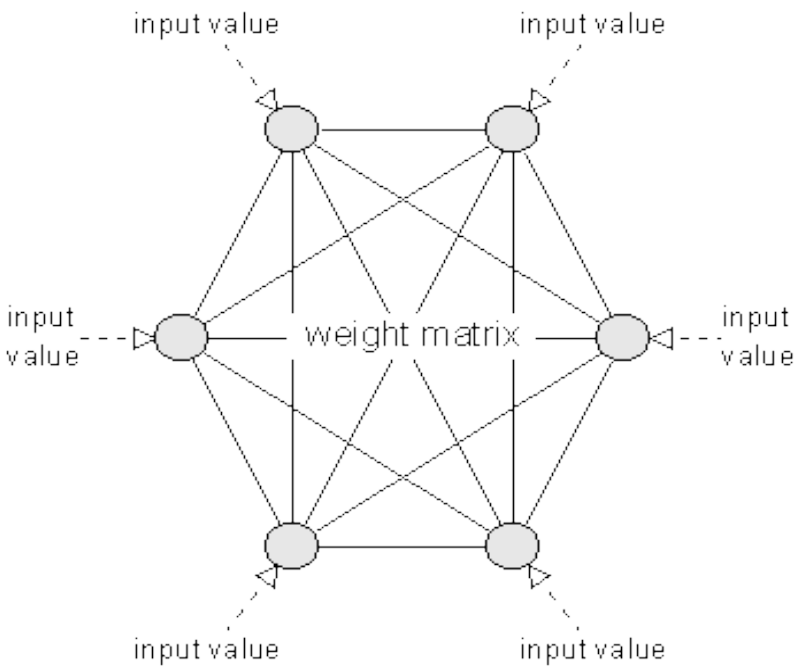
Three neuron layer Neural Net:



❑Neural nets are being constructed to solve problems that cannot be solved using conventional algorithms, such as optimization and classification problems:

→ pattern association
→ pattern classification
→ regularity detection
→ image processing
→ speech analysis
→ optimization problems
→ robot steering
→ processing of inaccurate or incomplete inputs
→ quality assurance
→ stock market forecasting …

Hopfield Model of Neural Networks



- The Hopfield model consists of a single layer of processing elements where each unit is connected to every other unit in the network other than itself.

- The connection weight matrix **W** of this type of network is square and symmetric, i.e., $W_{ij} = W_{ji}$ for i, j = 1, 2, ..., m. Each unit has an extra external input I_i. This extra input leads to a modification in the computation of the net input to the units:

$$\text{Input}_j = \sum_{i=1}^{m} x_i W_{ij} + I_j$$

- Discrete Hopfield Model:

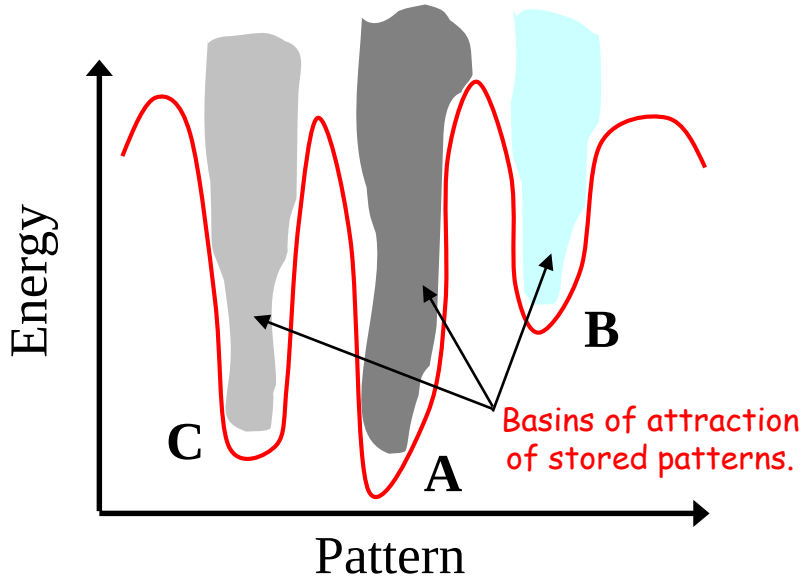
$$x_i(t+1) = \begin{cases} +1 & \text{if Input}_i > \theta_i \\ x_i(t) & \text{if Input}_i = \theta_i \\ -1 & \text{if Input}_i < \theta_i \end{cases}$$

$$E = -\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} x_i W_{ij} x_j - \sum_{i=1}^{m} x_i I_i - \sum_{i=1}^{m} x_i \theta_i$$

After many updates **discrete Hopfield Neural Net converges toward a local minimum of the energy function that corresponds to stored pattern** → The stored pattern to which the network converges depends on the input pattern and the connection weight matrix.

Hopfield Neural Net: Memory and Learning

- Content Addressable Memory has to be able to: Store, Recall, and Display Patterns



To store m patterns into the memory chose couplings according to:

$$W_{ij} = \frac{1}{M}\sum_m x_i(m)x_j(m)$$

If the number of stored patterns exceeds $0.13N$ the energy landscapes changed dramatically (stored patterns become unstable and system ceases to function as a memory) → **like phase transition in spin glasses**

- A new pattern p can be learned by adding a small contribution to interactions:

$$W_{ij}(\text{new}) = \beta W_{ij}(\text{old}) + \alpha x_i(p)x_j(p)$$

where $x_i(p)$ is the new pattern, α is parameter that controls how fast the learning should occur, and the value of β can be adjusted to allow for the fading of old memories.