Introduction to Deterministic Chaos

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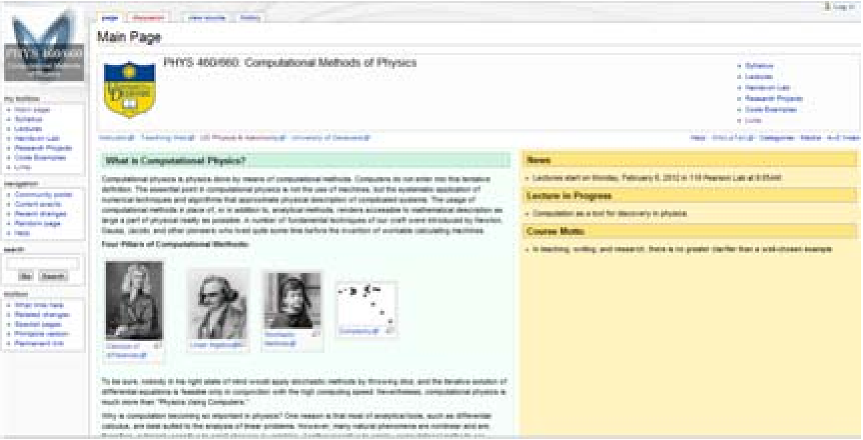
PHYS 460/660: Computational Methods of Physics

http://wiki.physics.udel.edu/phys660



Chaos vs. Randomness

Do not confuse **chaotic** with **random** temporal dynamics:

Random:

- irreproducible and unpredictable

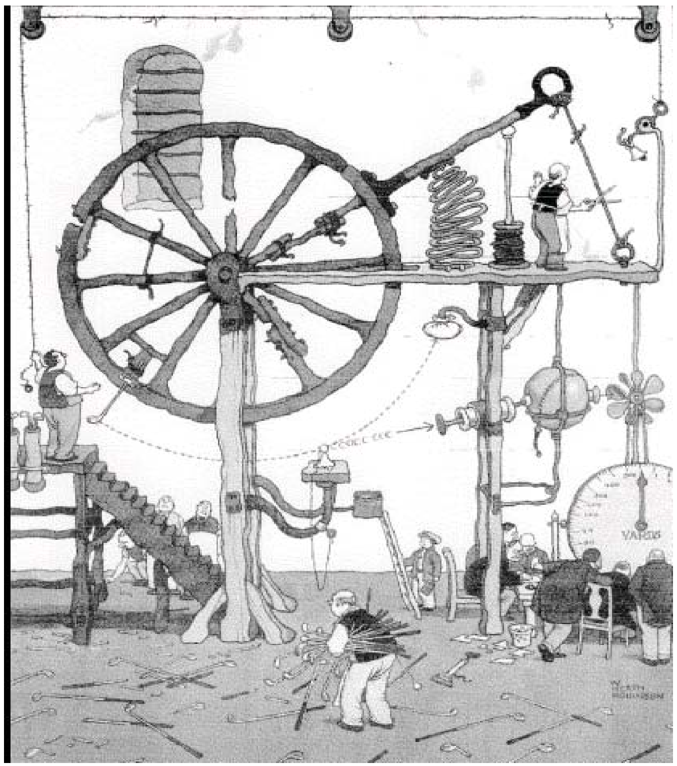
Chaotic (use characteristics below as definition):

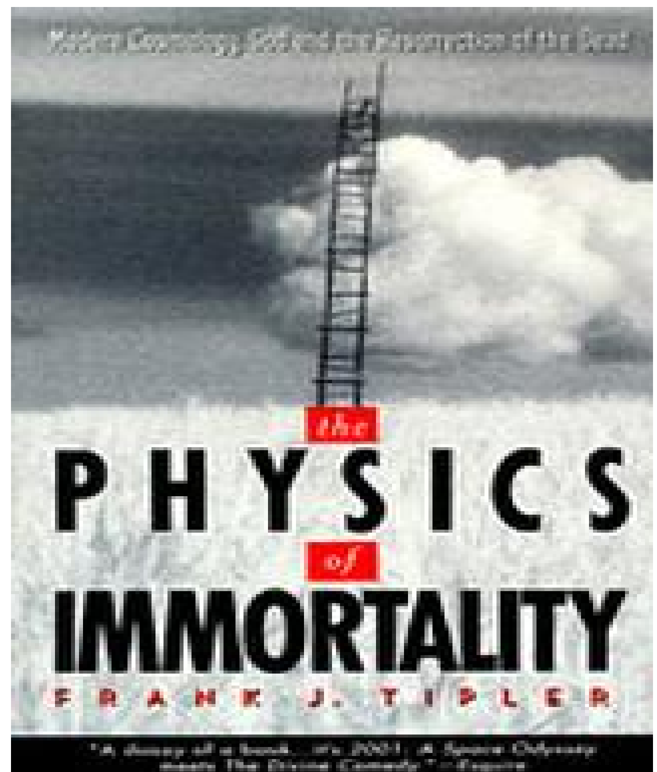
- **irregular** in time (it is not even the superposition of periodic motions – it is really aperiodic) for a simple system containing only few degrees of freedom
- **deterministic** - same initial conditions lead to same final state - but the final state is very different for small changes to initial conditions
- **difficult or impossible** to make long-term prediction!
- **complex, but ordered**, in phase space: it is associated with a fractal structure

Clockwork (**Newton**) vs. Chaotic (**Poincaré**) Universe

- Suppose the Universe is made of particles of matter interacting according to **Newton laws** → this is just a dynamical system governed by a (very large though) set of differential equations.

- Given the starting positions and velocities of all particles, there is a unique outcome → **P. Laplace's Clockwork Universe (XVIII Century)!**





Brief Chaotic History: 1892 - Poincaré invented Hamiltonian or Conservative Chaos

Henri Poincare

Birth of Chaos Theory

- In 1887 the King of Sweden offered a prize to the person who could answer the question "Is the solar system stable?"
- Poincare, a French mathematician, won the prize with his work on the three-body problem
- He considered, for example, just the Sun, Earth and Moon orbiting in a plane under their mutual gravitational attractions
- Like the pendulum, this system has some unstable solutions
- Introducing a Poincare section, he saw that homoclinic tangles must occur
- These would then give rise to chaos and unpredictability

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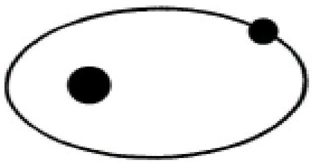
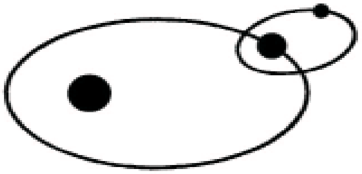


Newton solved the 2-body problem



Poincaré showed that the 3-body problem is essentially 'unsolvable'

Footnote: Did Poincaré get the money?

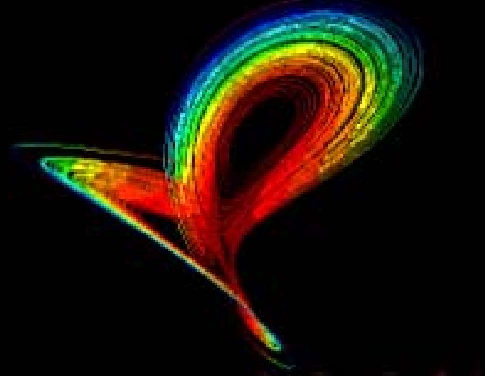
- **Jules Henri Poincaré** was dubbed by E. T. Bell as the **last universalist** — a man who is at ease in all branches of mathematics, both pure and applied — Poincaré was one of these rare savants who was able to make many major contributions to such diverse fields as analysis, algebra, topology, astronomy, and **theoretical physics**.

- While Poincaré did not succeed in giving a complete solution, **his work was so impressive that he was awarded the prize anyway**. The distinguished Weierstrass, who was one of the judges, said, "this work cannot indeed be considered as furnishing the complete solution of the question proposed, but that it is nevertheless of such importance that *its publication will inaugurate a new era in the history of celestial mechanics*." (a lively account of this event is given in *Newton's Clock: Chaos in Solar System*)

- **To show how visionary Poincaré was, it is perhaps best to read his description of the hallmark of chaos - sensitive dependence on initial conditions:**

"If we knew exactly the laws of nature and the situation of the universe at the initial moment, we could predict exactly the situation of that same universe at a succeeding moment. but even if it were the case that the natural laws had no longer any secret for us, we could still only know the initial situation *approximately*. If that enabled us to predict the succeeding situation with *the same approximation*, that is all we require, and we should say that the phenomenon had been predicted, that it is governed by laws. But it is not always so; it may happen that **small differences in the initial conditions produce very great ones in the final phenomena**. A small error in the former will produce an enormous error in the latter. Prediction becomes impossible, and we have the fortuitous phenomenon. - in a 1903 essay "Science and Method"

Brief Chaotic History: 1963 Lorentz Discovers Dissipative Chaos in Numerical Solution to ODE



- In 1963 Lorenz was trying to improve weather forecasting
- Using a computer, he discovered the first chaotic attractor
- Three variables (*x*, *y*, *z*) define convection of the atmosphere
- Changing in time, these variables give a trajectory in a 3D space
- From all starts, trajectories settle onto a strange, chaotic attractor
- Right and left flips occur as randomly as heads and tails
- Prediction is impossible

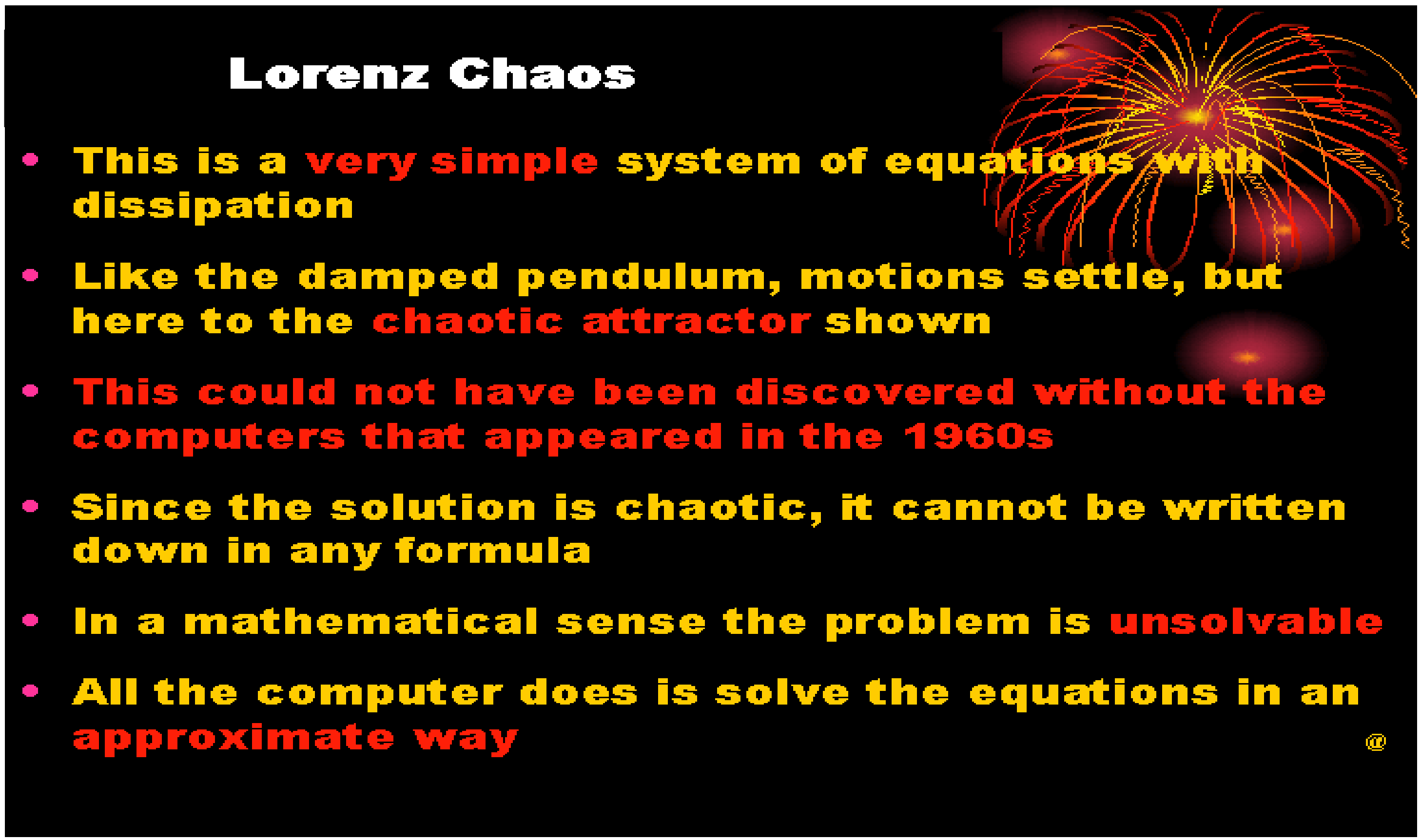
$$x' = -10\,(x - y)$$

$$y' = 28\,x - y - x\,z$$

$$z' = x\,y - (8/3)\,z$$

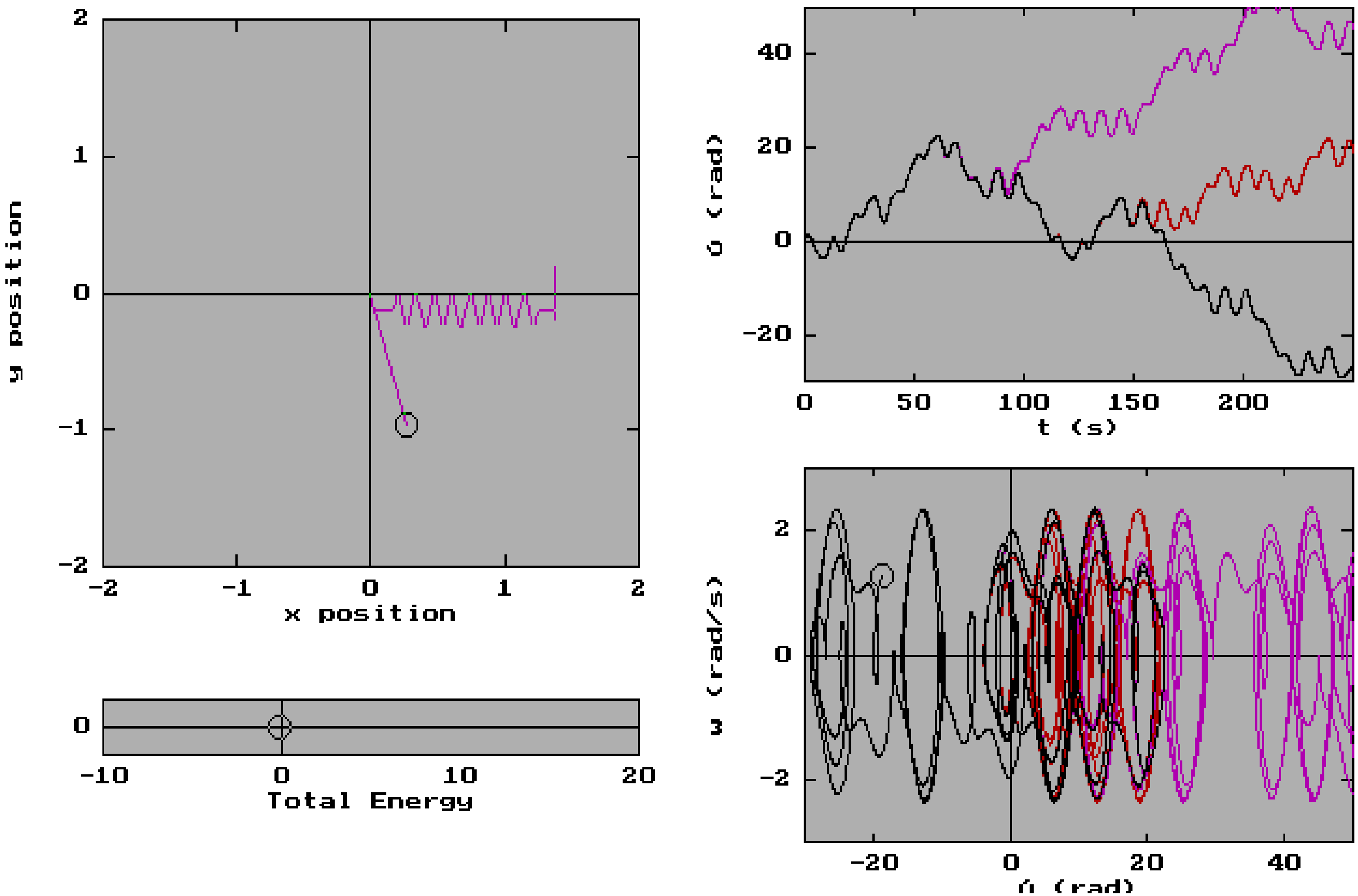
Chaos in the Brave New World of Computers

Poincaré created an original method to understand chaotic systems, and discovered their very complicated time evolution, but:
"It is so complicated that I cannot even draw the figure."



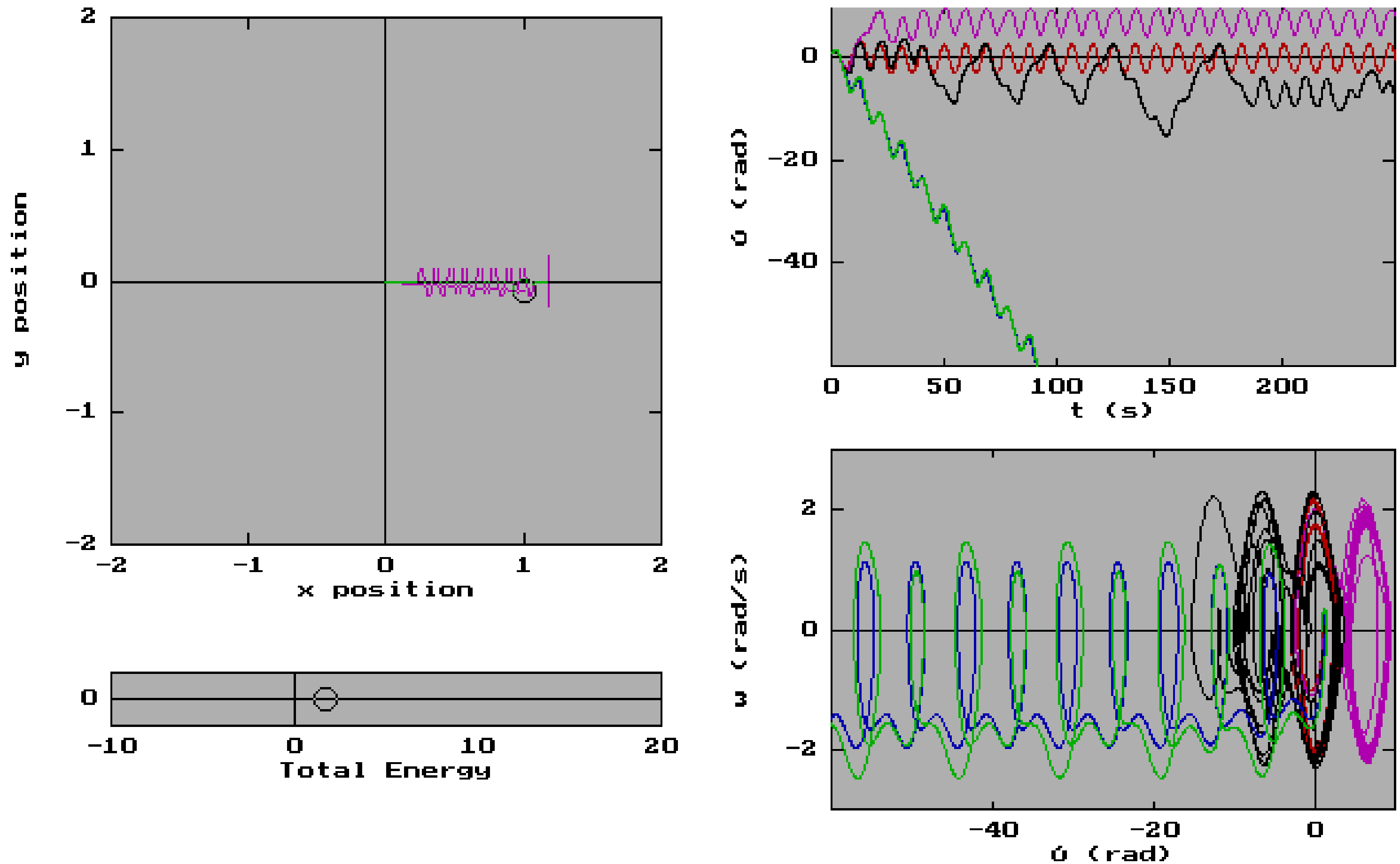
Example: Damped Driven Pendulum

- Initial position (i.e., angle) is at 1, 1.001, and 1.000001 rad:



Tuning the driving force and transition to chaos

- Not every dumpled driven pendulum is chaotic → depends on the driving force: f = 1, 1.07, 1.15, 1.35, 1.45



Can Chaos Be Exploited?

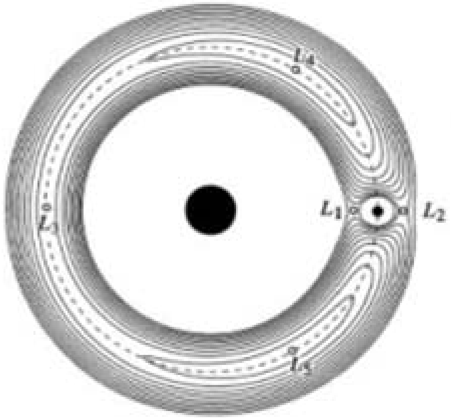
Using chaos today

The rich dynamics of a chaotic state often allow it to be easily controlled

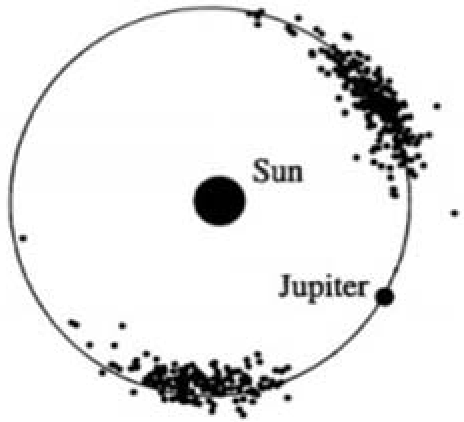
To get a driven pendulum spinning quickly, either way, it is best to keep it in its chaotic state

Chaos is used by rocket scientists to minimise the fuel needed for a mission @

Lagrange points and zero-velocity curves (mass ratio ≈ Earth-Moon)



Distribution of asteroids near the orbit of Jupiter



USING CHAOS FOR SPACE FLIGHT

Consider a **rotating** reference frame in which the Sun and Earth appear stationary.

A spacecraft can remain stationary at 5 points, named after Lagrange.

Points L_1, L_2 and L_3 lie on the **Sun-Earth axis** and are **unstable** equilibrium states.

The *SOHO* **spacecraft** was maintained in a **halo orbit** around L_1 to observe the sun.

Points L_4 and L_5 are the **triangular points**, and in the solar system most are **stable**.

Some **asteroids** cluster around the triangular points in the **Sun-Jupiter system**.

GENESIS uses a **chaotic orbit** between L_1 and L_2. Almost coasting, **it uses little fuel!**

The Japanese *HITEN* rescue mission used a chaotic **Earth-Moon** trajectory.

USING CHAOS TO SAVE THE EARTH

Long age, an **asteroid crashed into the Earth** and killed all the **dinosaurs**.

It **could happen again**, and destroy all life on Earth.

An asteroid on a collision course is most easily deflected while **in a chaotic region**.

Examples of Chaos in Physical Systems

- Chaos is seen in many physical systems:
  - fluid dynamics (weather patterns) and turbulence
  - some chemical reactions
  - Lasers
  - electronic circuits
  - particle accelerators
  - plasma (such as in fusion reactors and space)

- **Conditions necessary for chaos:**
  - system has 3 independent dynamical variables
  - the equations of motion are non-linear

Concepts in Dynamical System Theory

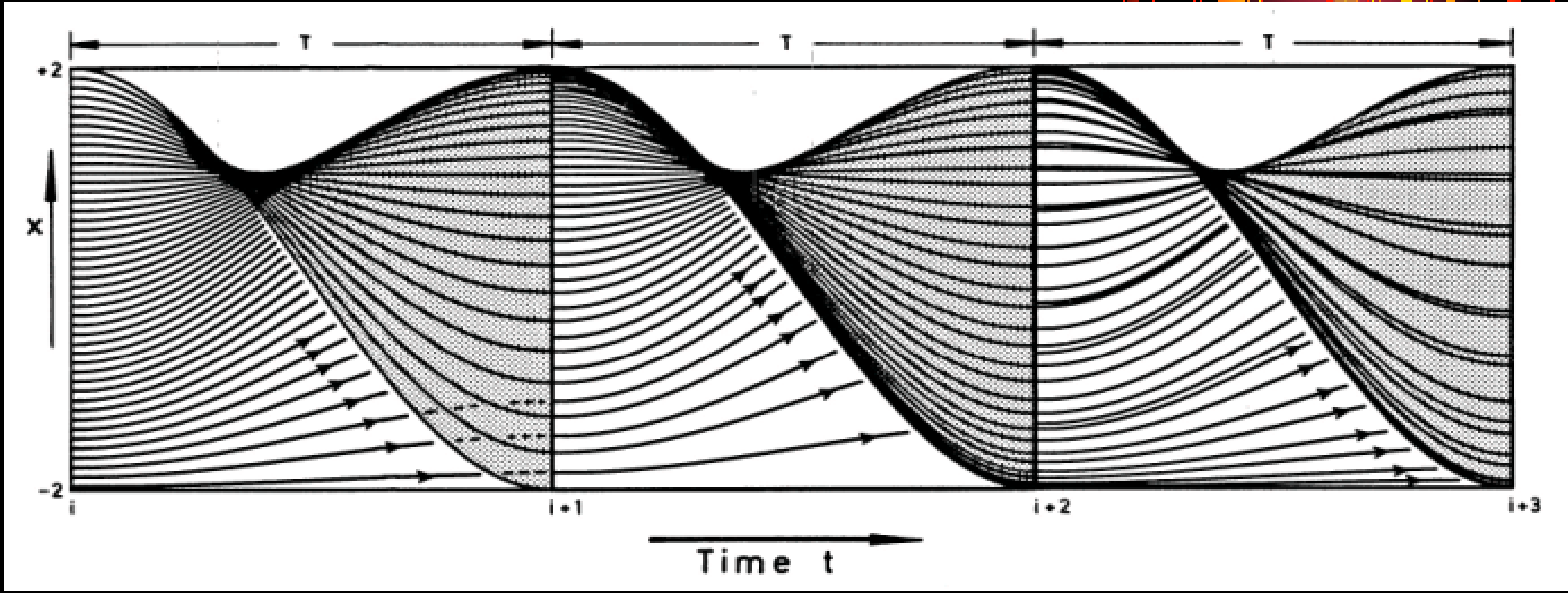
- A dynamical system is defined as a deterministic mathematical prescription for evolving the state of a system forward in time.
- **Example: A system of N first-order and autonomous ODE:**

$$\left.\begin{aligned}\frac{dx_1}{dt} &= F(x_1, x_2, \ldots, x_n)\\ \frac{dx_2}{dt} &= F(x_1, x_2, \ldots, x_n)\\ &\vdots\\ \frac{dx_N}{dt} &= F(x_1, x_2, \ldots, x_n)\end{aligned}\right\} \Rightarrow \begin{cases}\text{set of points } (x_1, x_2, \ldots, x_n) \text{ is phase space}\\ \left[x_1(t), x_2(t), \ldots, x_n(t)\right] \text{ is trajectory or flow}\end{cases}$$

$$\boxed{N \geq 3 + \text{nonlinearity} \Rightarrow \text{CHAOS becomes possible !}}$$

Why Nonlinearity and at Least 3D Phase Space?

Folding and Mixing



- There is no crossing in phase space: so how do complex chaotic motions arise?
- The answer is by divergence, folding and mixing (possible with **nonlinearity** and **3D**) @

Ordinary Differential Equation for Damped Driven Pendulum

- Nonlinear ODE of the second order:

$$ml\frac{d^2\theta}{dt^2} + c\frac{d\theta}{dt} + mg\sin\theta = A\cos(\omega_D t + \phi)$$

- First step for computational approach → convert ODE into a **dimensionless** form:

$$\frac{d\omega}{dt} + q\frac{d\theta}{dt} + \sin\theta = f_0\cos(\omega_D t + \phi)$$

How to Prepare Equations in Dimensionless Form Before Putting Them Into the Computer

1. Introduce dimensionless space and time (x', t') coordinates via:

$$x = Lx' \qquad t = Tt'$$

2. Switch to dimensionless velocity and acceleration:

$$\frac{dx}{dt} = \frac{L}{T}\frac{dx'}{dt'} \qquad \frac{d^2x'}{dt'^2} = \frac{T^2}{L} f\left(Lx', \frac{L}{T}\dot{x}', Tt'; \text{parameters}\right)$$

and choose L and T (natural length and time scale of the system), so that parameter dependence is simplest (i.e., wherever possible the prefactors should be 1).

Example: $$\ddot{\theta} \equiv \frac{d^2\theta}{dt'^2} = -\frac{T^2 g}{L}\sin\theta \overset{T=\sqrt{L/g}}{\Longrightarrow} \ddot{\theta} + \sin\theta = 0$$

Where is Nonlinearity and ≥Three Dynamical Variables in Damped Driven Pendulum?

$$\frac{d\omega}{dt} + q\frac{d\theta}{dt} + \sin\theta = f_0\cos(\omega_D t)$$

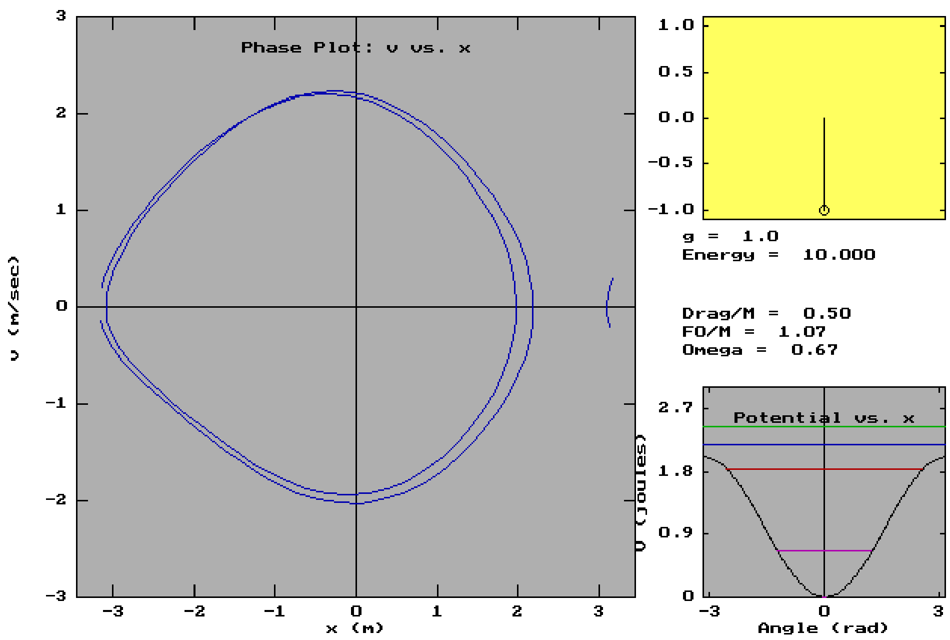
- Non-linear term: $\sin\theta$
- Three dynamic variables: ω, θ, t

$$\left.\begin{aligned} x_1 &= \frac{d\theta}{dt} \\ x_2 &= \theta \\ x_3 &= \omega_D t \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} \frac{dx_1}{dt} &= f_0\cos x_3 - \sin x_2 - q x_1 \\ \frac{dx_2}{dt} &= x_1 \\ \frac{dx_3}{dt} &= \omega_D \end{aligned}\right.$$

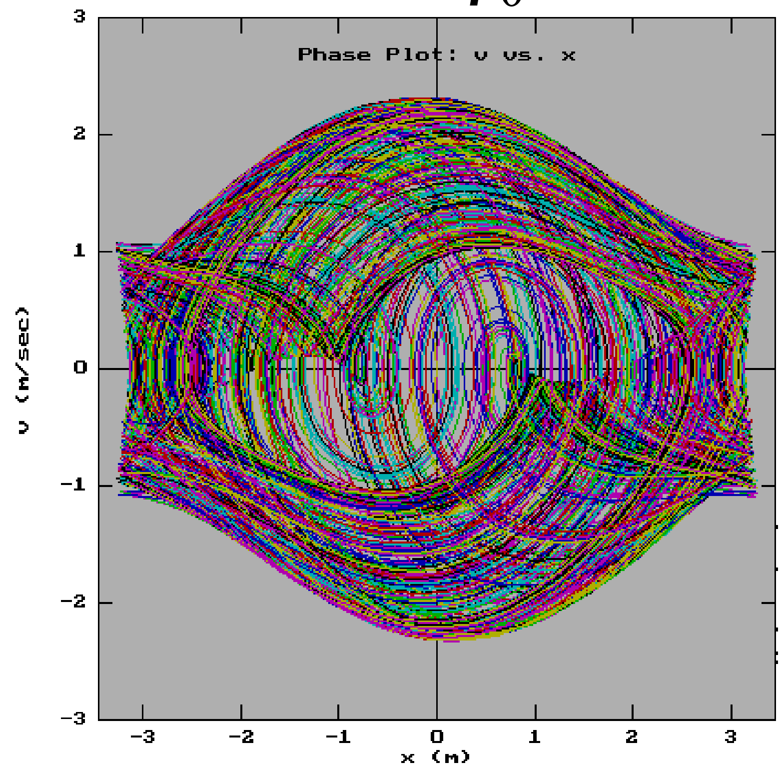
- This system is chaotic only for certain values of q, f_0, ω_D
- In the examples we use $q = 1/2, \omega_D = 2/3,\ f_0 \in (1,2)$

Routes to Chaos: Period Doubling

$$f_0 = 1.07$$

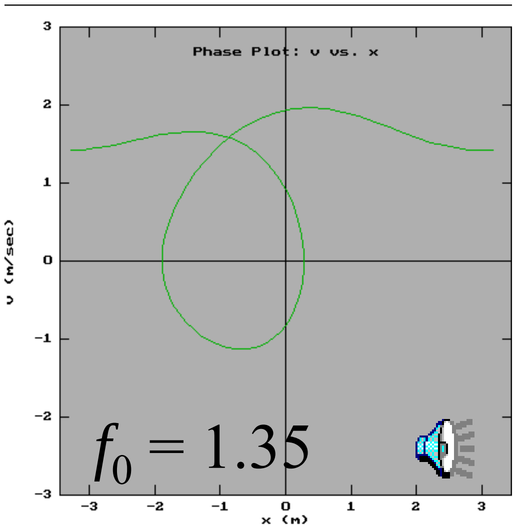


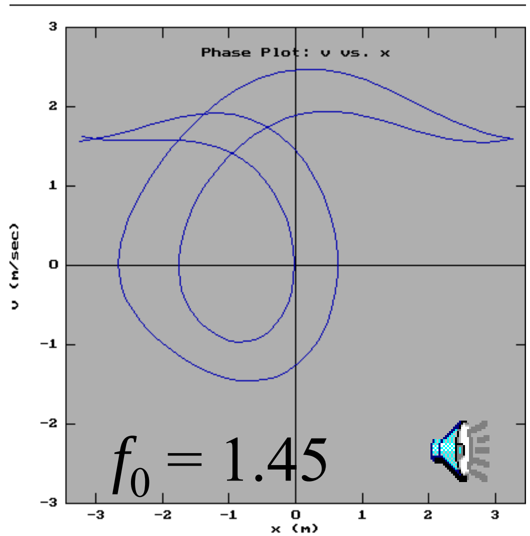
$$f_0 = 1.15$$

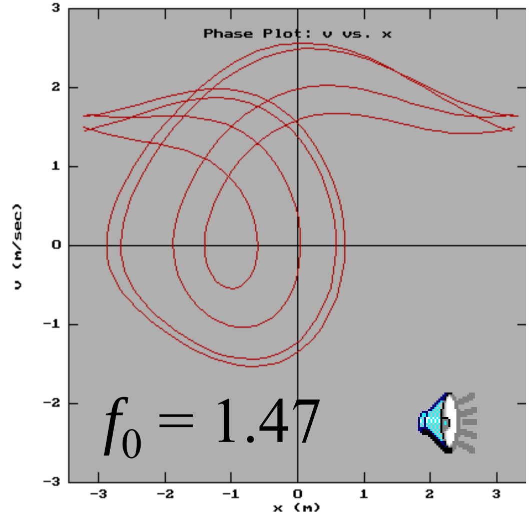


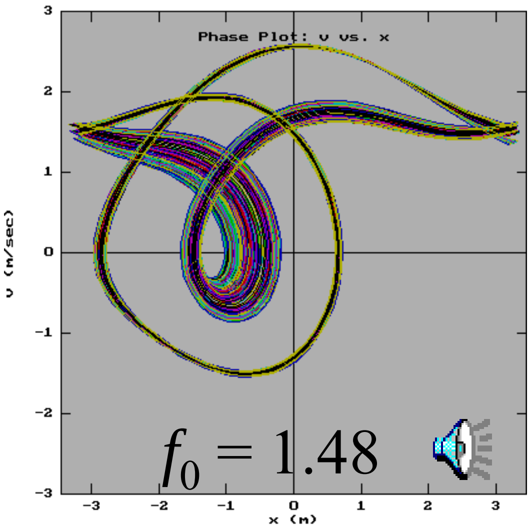
- To watch the onset of chaos (as f_0 is increased) we look at the motion of the system in **phase space**, once transients die away
- Pay close attention to the period doubling that precedes the onset of chaos.

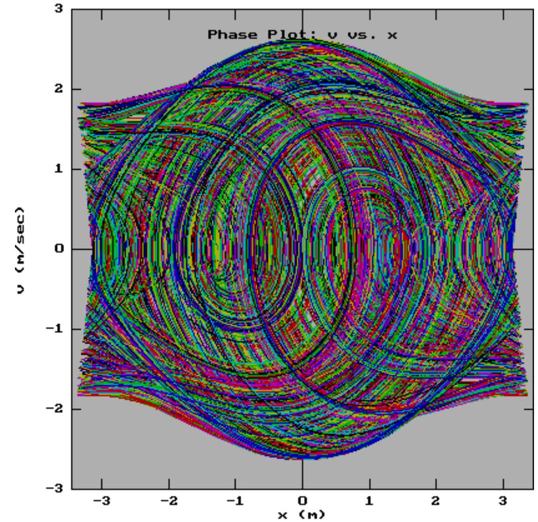
The "Sound" of Chaos











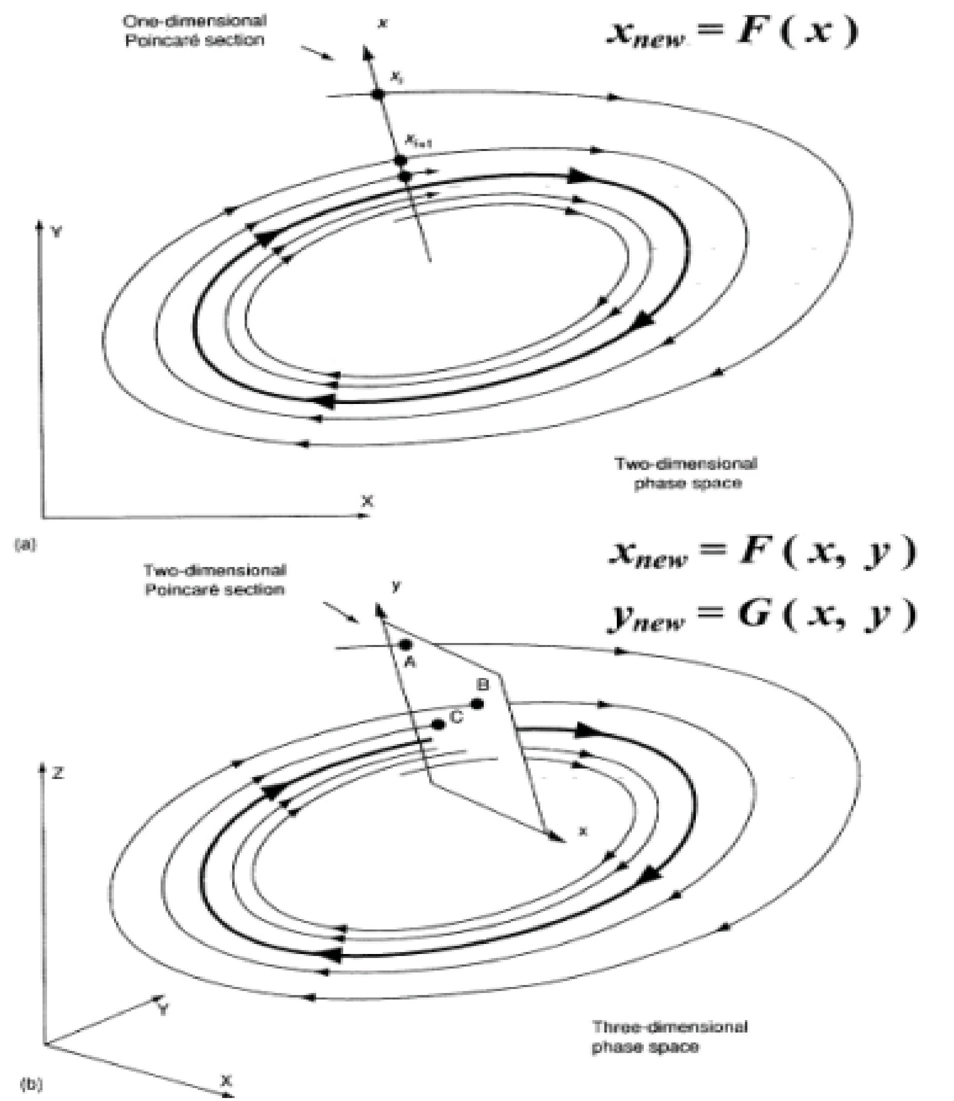
$f_0 = 1.50$

Deterministic Chaos Look Like Random Motion In Real Space → Order Emerges in Phase Space

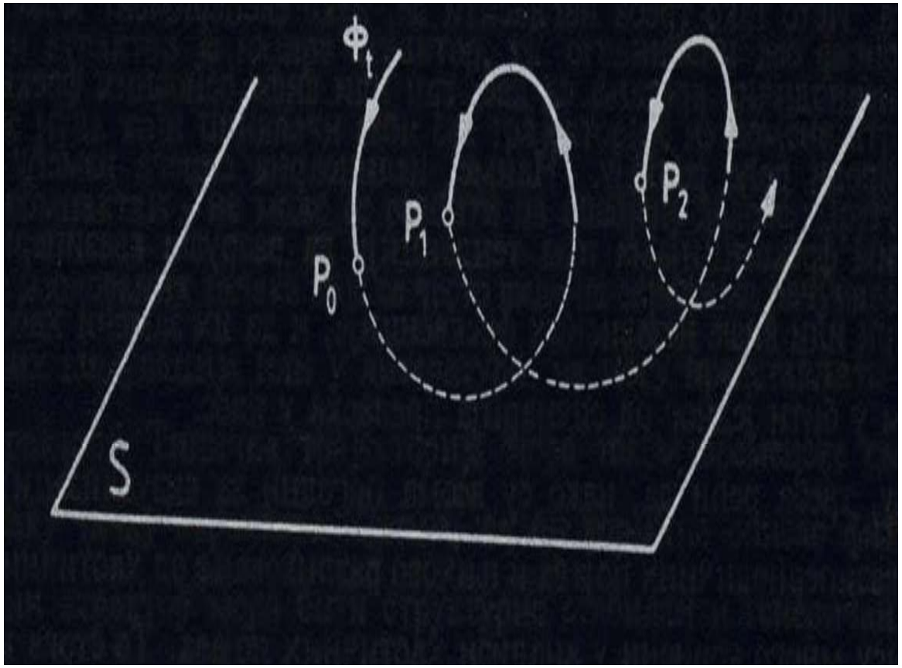
- Strange attractors in **Dissipative Chaos**
- K[olmogorov]A[rnold]M[oser] torus of regular motion becomes deconstructed in **Conservative Chaos** which is then characterized by chaotic bands and regular islands
- Poincaré sections probe fractals structures in phase space generated by dissipative or conservative chaos
- Lyapunov exponents and Kolmogorov entropy
- Fourier spectrum and autocorrelation functions

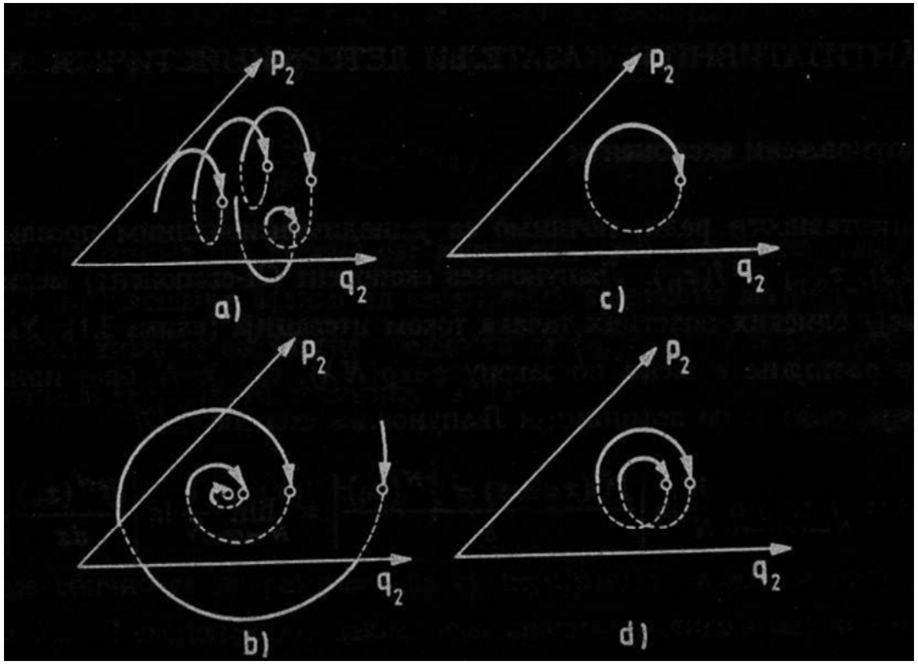
Poincaré Section

- To examine chaos, Poincare used the idea of a section
- This cuts across the phase-space orbits
- The original system flows in continuous time
- On the section, we observe steps in discrete time
- The flow is replaced by what is called an iterated map
- The dimension of the phase-space is reduced by one @



Examples of Poincaré Section

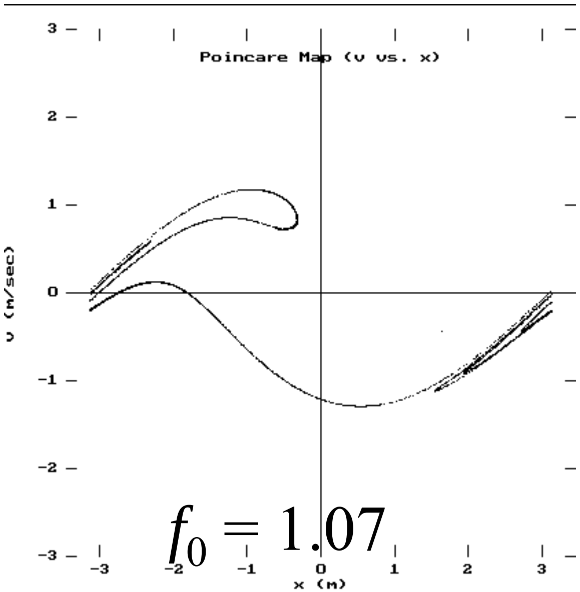




Poincare Map: Continuous time evolution is replace by a discrete map

$$\boxed{P_{n+1} = f_P(P_n)}$$

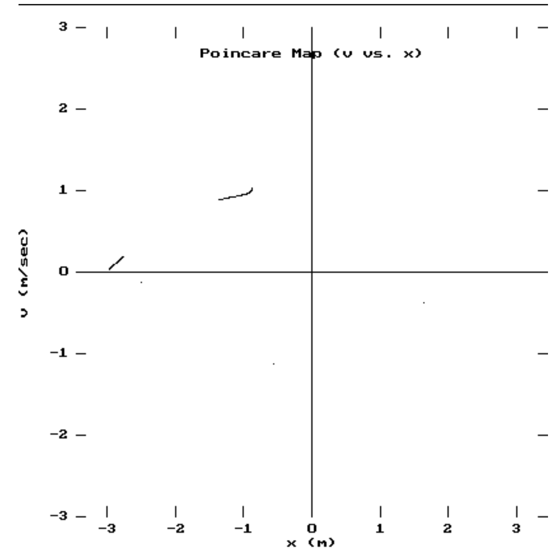
Poincaré Section of Damped Driven Pendulum: A Slice of the 3D Phase Space at a Fixed Value of $\omega_D t \bmod 2\pi$

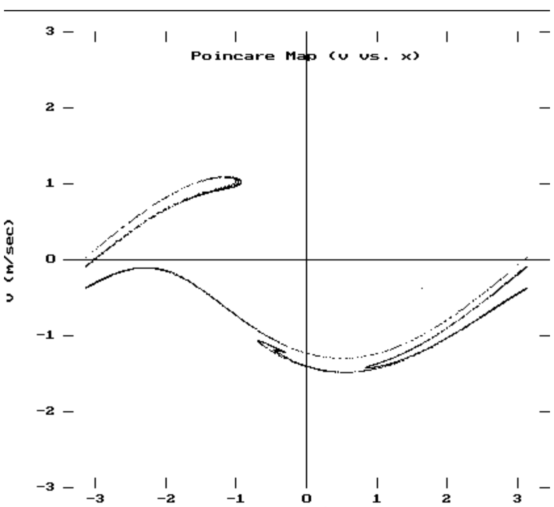


$f_0 = 1.07$

$q = 0.25$

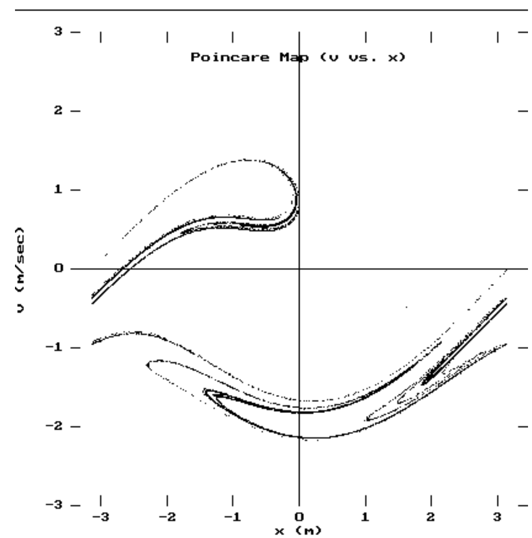
$f_0 = 1.48$





$f_0 = 1.50$

$f_0 = 1.15$

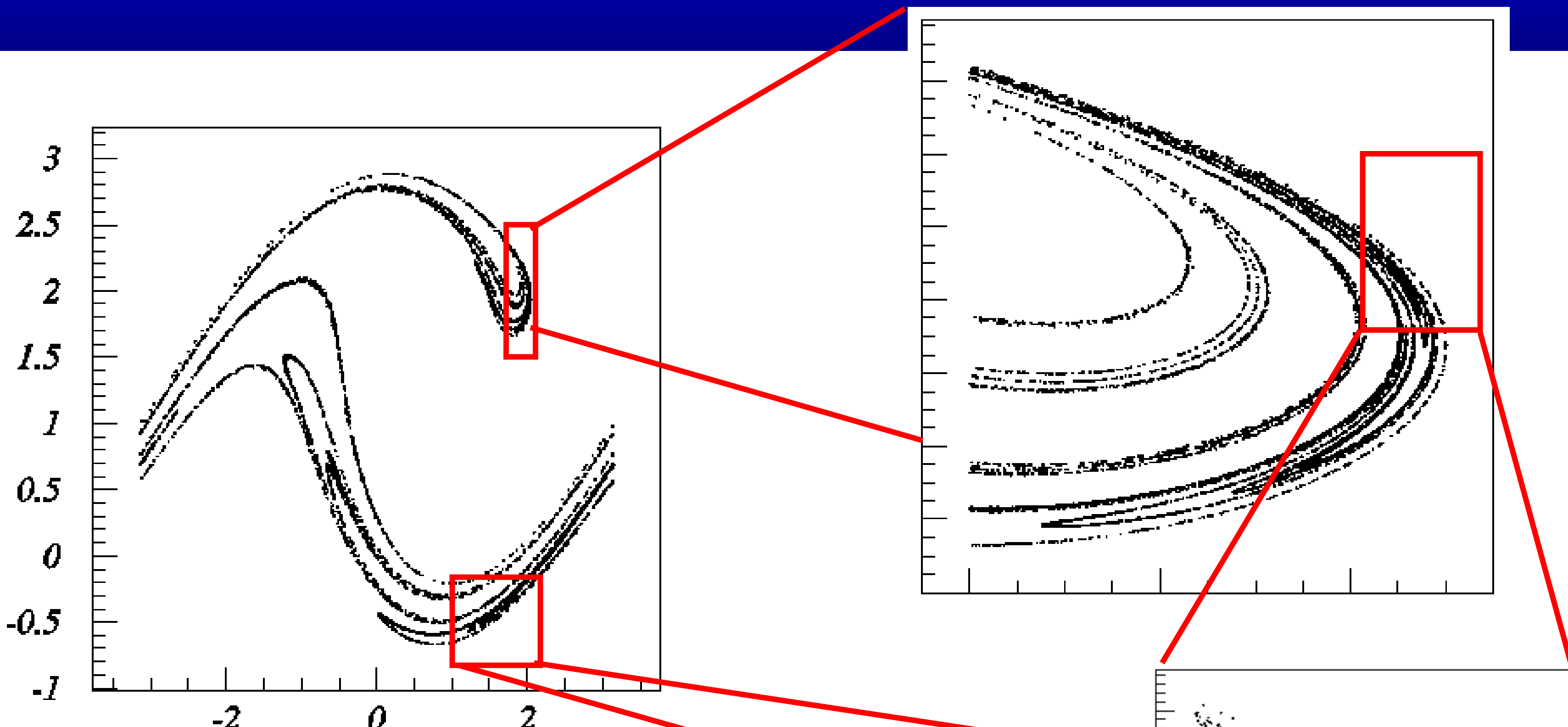


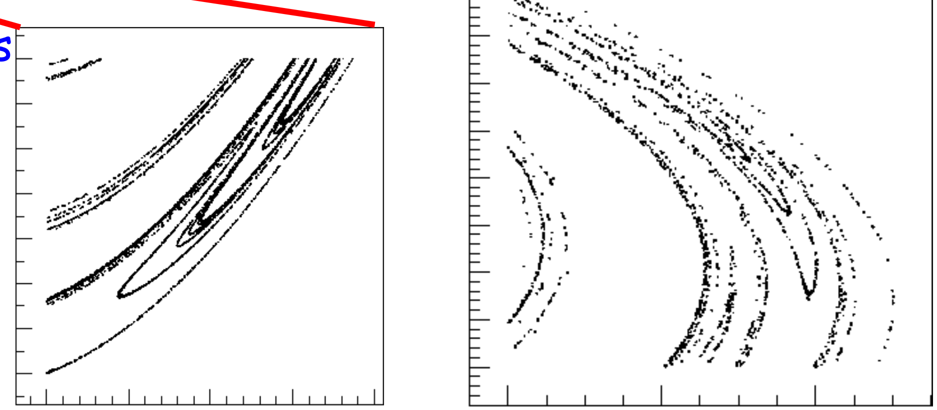
Attractors in Phase Space

- The surfaces in phase space which the pendulum follows, after transient motion decays, are called attractors.

- Non-Chaotic Attractor Examples:
  - →for a damped undriven pendulum, attractor is just a point at $\theta=\omega=0$ (0D in 2D phase space).
  - →for an undamped pendulum, attractor is a curve (1D attractor).

Fractal Nature of Strange Attractors



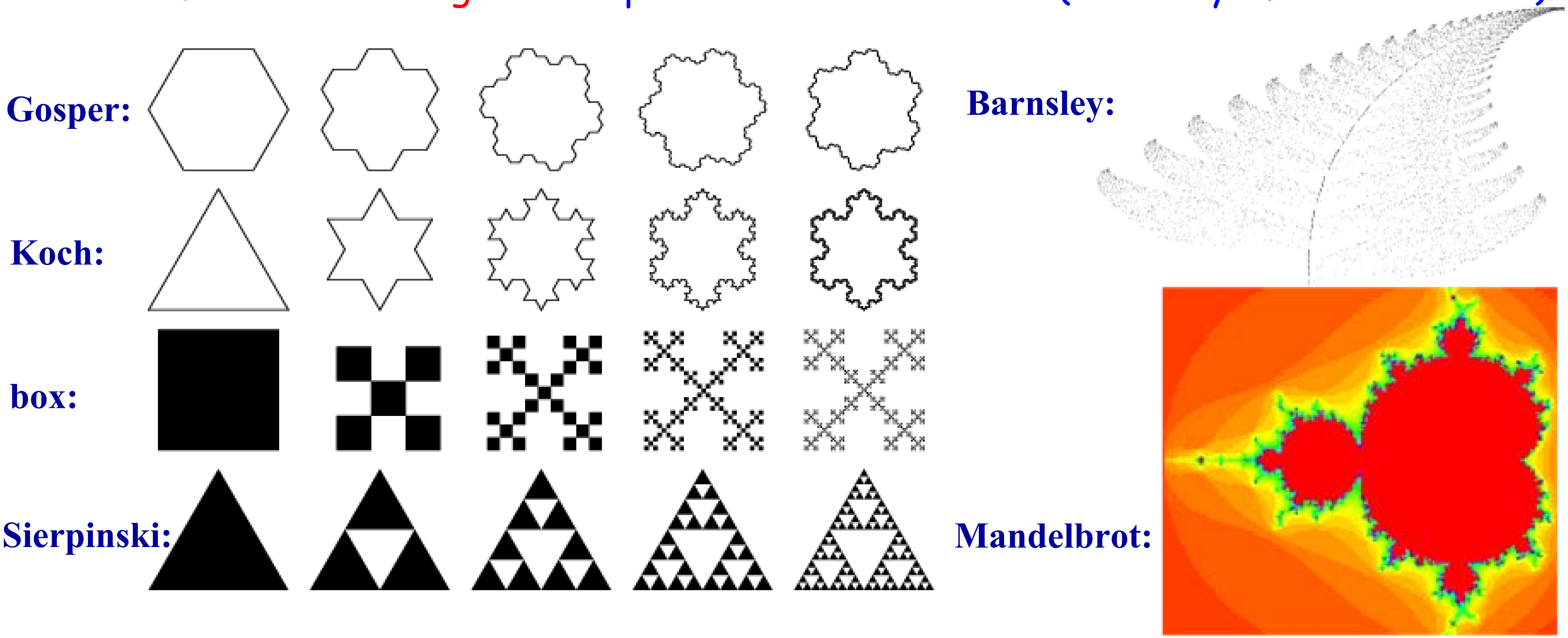


- Chaotic attractors of dissipative systems are **strange** → fractals with non-integer dimension (2<dim<3 for pendulum) and zero volume.
- The fine structure is quite complex and similar to the gross structure – fractals reveal **self-similarity** when viewed by a magnifying glass.

World of Fractals in Pictures

- A fractal is an object or quantity that displays self-similarity on all scales - the object need not exhibit exactly the same structure at all scales, but the same "type" of structures must appear on all scales.

- Their surface area is large and depends on the resolution (accuracy of measurement).

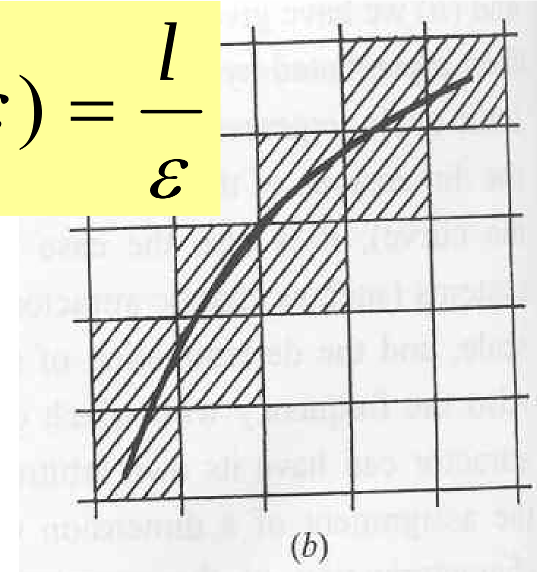


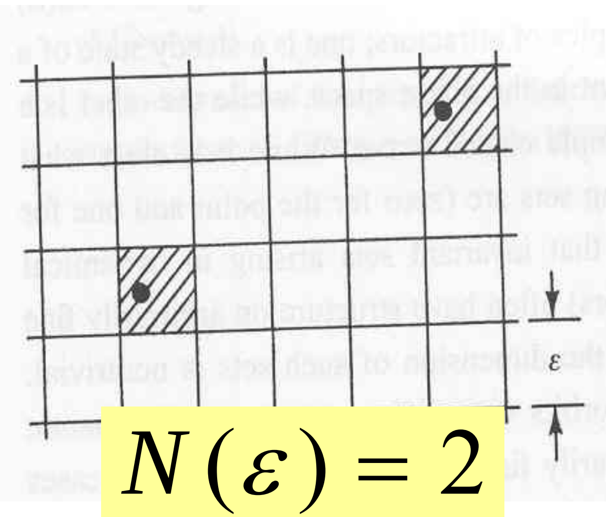
- The prototypical example for a fractal in nature is the length of a coastline measured with different length rulers. The shorter the ruler, the longer the length measured, a paradox known as the coastline paradox.

How Do We Define Dimension of a Geometrical Object?

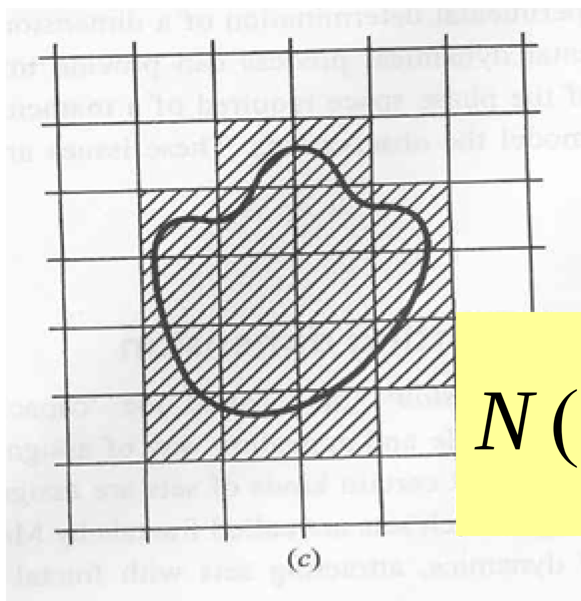
$$N(\varepsilon) \sim \varepsilon^{-D_0} \Rightarrow D_0 = \frac{\ln N(\varepsilon)}{\ln 1/\varepsilon},\ \varepsilon \ll 1$$

$$N(\varepsilon) = \frac{l}{\varepsilon}$$





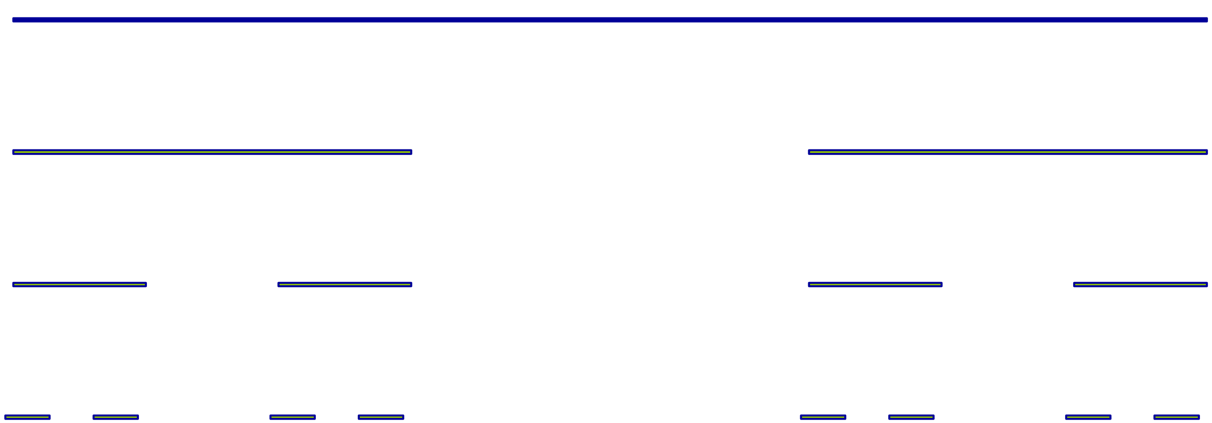
$$N(\varepsilon) = 2$$



$$N(\varepsilon) = \frac{A}{\varepsilon^2}$$

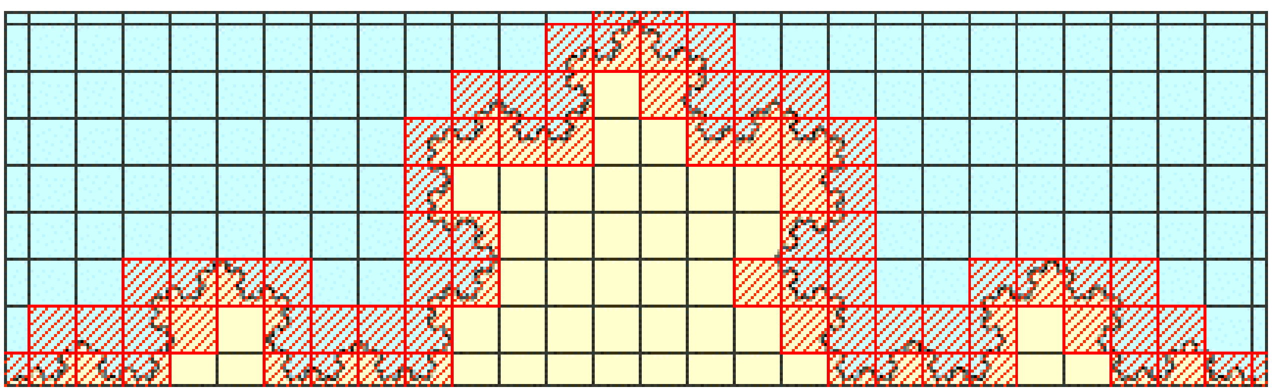
Non-Trivial Examples: Cantor Set and Koch Curve

- The Cantor set is produced as follows:



N	ε
1	1
2	1/3
4	1/9
8	1/27

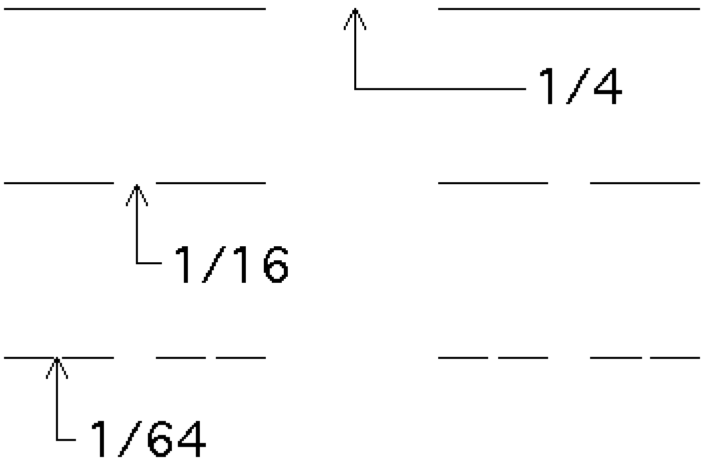
$$2^n \text{ boxes of size } \varepsilon = \left(\frac{1}{3}\right)^n \Rightarrow N(\varepsilon) = \varepsilon^{-\ln 2/\ln 3} \Rightarrow D_0 = \frac{\ln 2}{\ln 3} = 0.576 < 1$$



- Koch curve

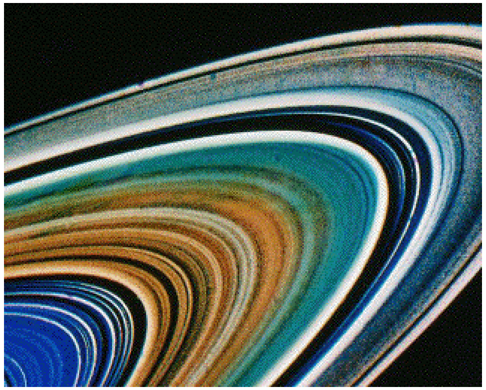
$$D_0 = \frac{\ln 4}{\ln 3}$$

Fat Fractals vs. Thin Fractals



- **Fat Cantor set**: we use the same procedure but vary the sizes of the pieces removed - first, remove the middle 1/4 of the unit interval → from each of the remaining two pieces remove an interval of length 1/16 = 1/42 → from each of the remaining four pieces remove an interval of length 1/64 = 1/43, and so on …

- Mandelbrot's conjecture: Radial cross-sections of Saturn's rings are fat Cantor sets:



- Fat fractals have non-zero volume and dimension →a measurable property of fat fractals is that their observed volume depends on the resolution in such a way that deviation from the exact volume decreases slowly and proportionally to a power of the resolution

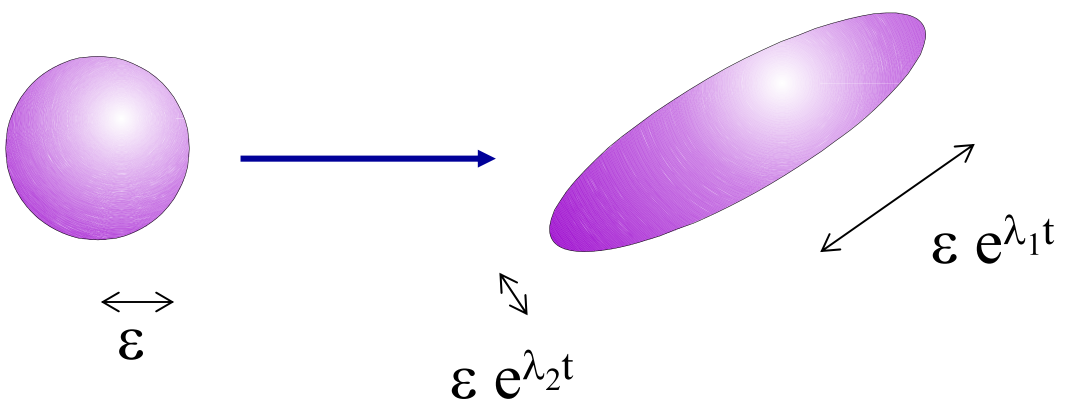
$$V(\varepsilon) - V \sim \varepsilon^{\alpha}$$

$$\alpha \equiv \text{fat fractal exponent}$$

- Standard thin fractals have vanishing volume $V(\varepsilon) = \varepsilon^d N(\varepsilon) \sim \varepsilon^{d-D_0} \ (\varepsilon \ll 1)$ (they are not space filling) and non-integer dimension $D_0 < d$, but their observed surface may tend to infinity.

Lyapunov Exponents

- The fractional dimension of a chaotic attractor is a result of the extreme sensitivity to initial conditions.
- **Lyapunov exponents** are a measure of the average rate of divergence of neighboring trajectories on an attractor.
- Consider a small sphere in phase space containing initial conditions → after a short time the sphere will evolve into an ellipsoid:



Connection Between Lyapunov Exponents and Fractal Dimension of Strange Attractors

- The average rate of expansion along the principle axes are the **Lyapunov exponents**
- Chaos implies that **at least one Lyapunov exponents is > 0!**
- For damped driven pendulum: $\sum_i \lambda_i = -q$ (sum of Lyapunov exponents is negative damping coefficient)
  - →no contraction or expansion along *t* direction, so that exponent is zero
  - →it can be shown that the dimension of the attractor is:

$$D_0 = 2 - \lambda_1 / \lambda_2$$

Lyapunov Exponents for Dissipative vs. Conservative Chaos

- For **Hamiltonian systems**, the Lyapunov exponents exist in additive inverse pairs, while one of them is always 0.

- In dissipative systems in an arbitrary *n*-dimensional phase space, there must always be one Lyapunov exponent equal to 0, since a perturbation along the path results in no divergence.

$\rightarrow$ (-, -, -, -, ...) fixed point (0-D)

$\rightarrow$ (0, -, -, -, ...) limit cycle (1-D)

$\rightarrow$ (0, 0, -, -, ...) 2-torus (2-D)

$\rightarrow$ (0, 0, 0, -, ...) 3-torus, etc. (3-D, etc.)

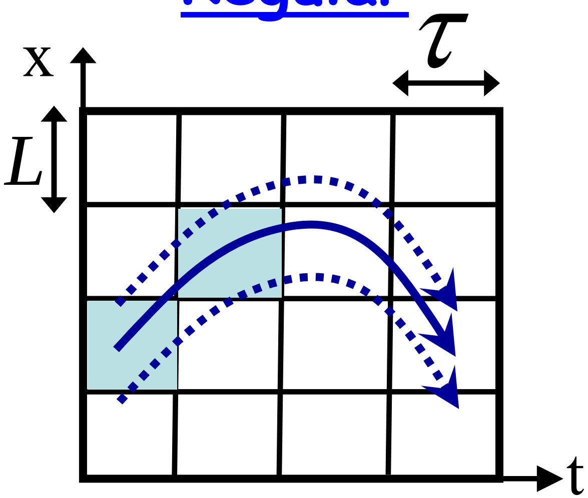
$\rightarrow$ (+, 0, -, -, ...) strange (chaotic) (2+-D)

$\rightarrow$ (+, +, 0, -, ...) hyperchaos, etc. (3+-D)

Kolmogorov Entropy

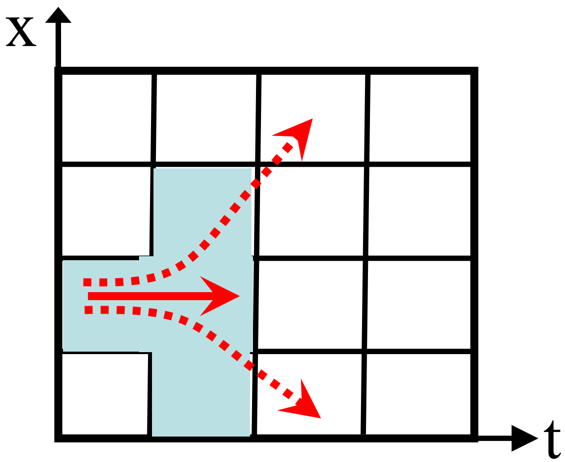
- **Interpretation:** Measures amount of information required to specify trajectory of a system in the phase space.
- **Alternatively Interpretation:** Measures rate at which initial information about the state of the system in phase space is washed out!

Regular



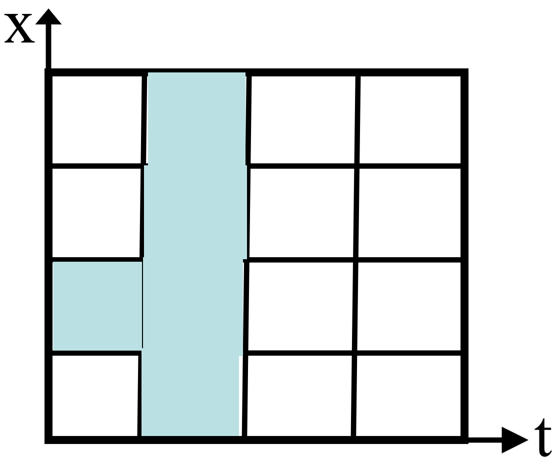
$$N_{cell} = 1 \Rightarrow K \approx \ln N_{cell} = 0$$

Chaotic



$$N_{cell} = e^{\lambda} \Rightarrow K \approx \ln N_{cell} = \lambda$$

Random



$$N_{cell} \to \infty \Rightarrow K \to \infty$$

Computer Simulations of Chaos in Damped Driven Pendulum

- From differential to difference equations – use Euler-Cromer method:

$$\left.\begin{aligned}\frac{d\omega}{dt} &= -\Omega^2\sin\theta - q\omega + f_D\sin t\\ \frac{d\theta}{dt} &= \omega\end{aligned}\right\} \Rightarrow \begin{cases}\omega_{n+1} = \omega_n - \Delta t\left(\Omega^2\sin\theta_n + q\omega_n - f_D\sin t\right)\\ \theta_{n+1} = \theta_n + \Delta t\,\omega_{n+1}\\ \theta_0 = \dfrac{\pi}{2}, \quad \omega_0 = 0\end{cases}$$

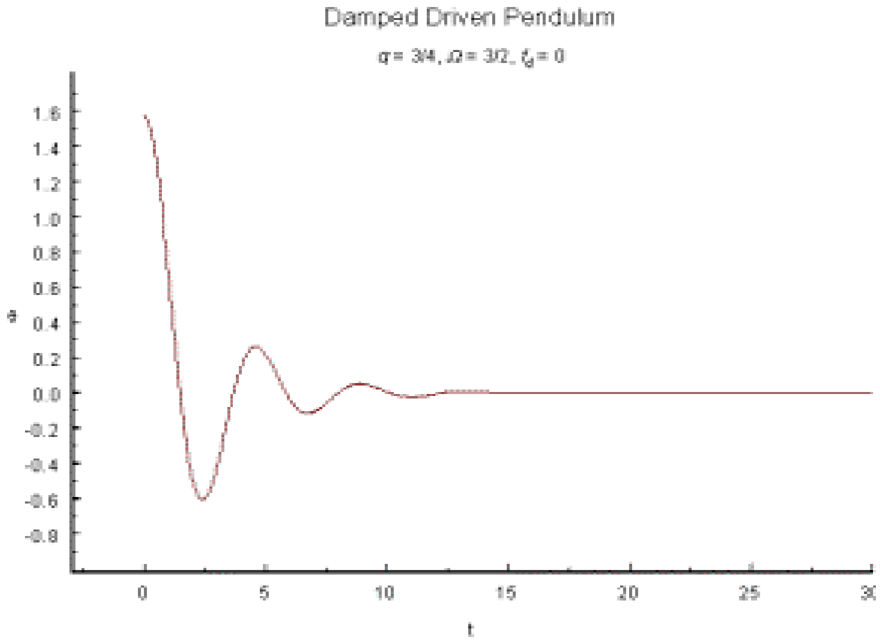
- Search the phase space to find aperiodic motion confined to strange attractors which fill densely Poicaré sections

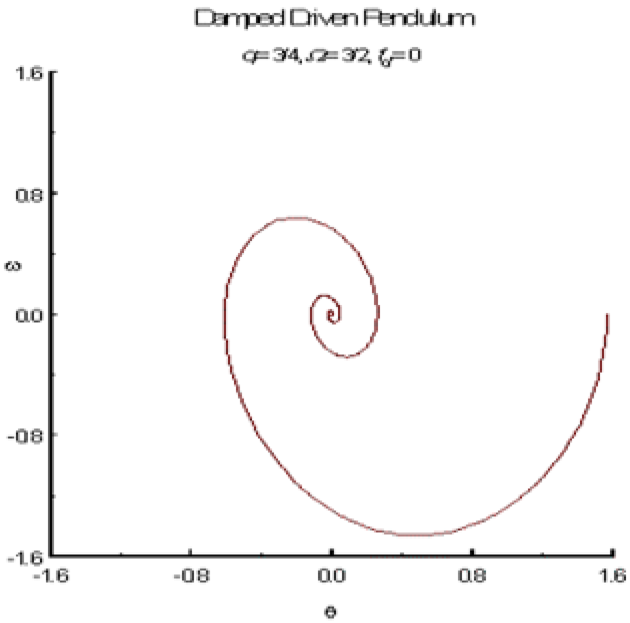
- Compute autocorrelation function to see if it drops to zero, while power spectrum (which is its Fourier transform) exhibits continuum of frequencies

$$x(\omega) = \int_0^\infty e^{i\omega t} x(t)\,dt \Rightarrow P(\omega) = |\,x(\omega)\,|^2$$

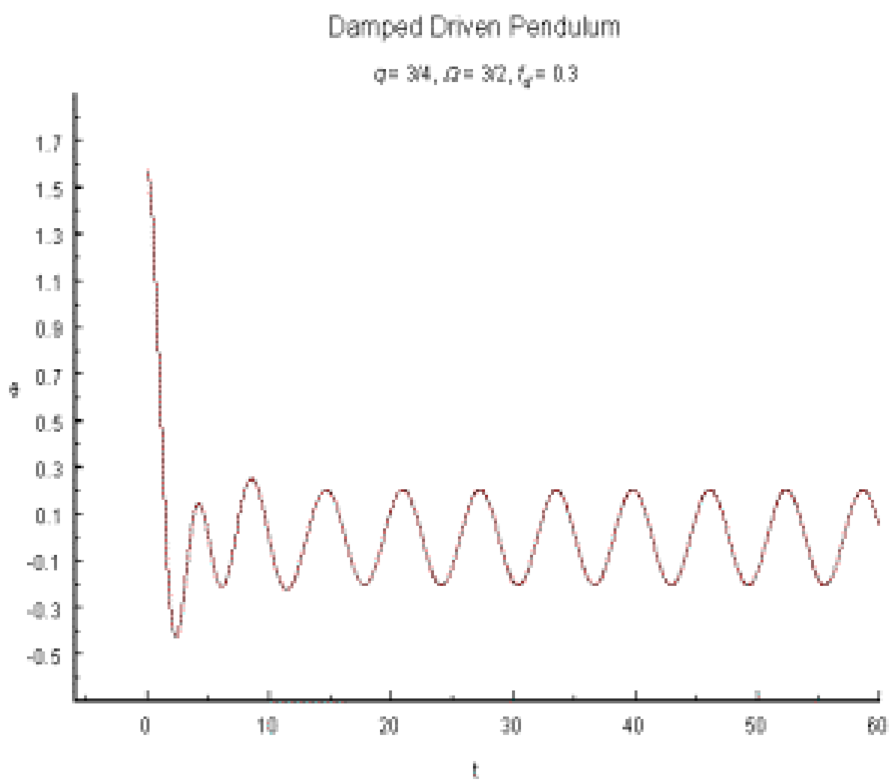
$$C(\tau) = \int_0^\infty \left[(x(t) - \overline{x}) \cdot (x(t+\tau) - \overline{x})\right] dt$$

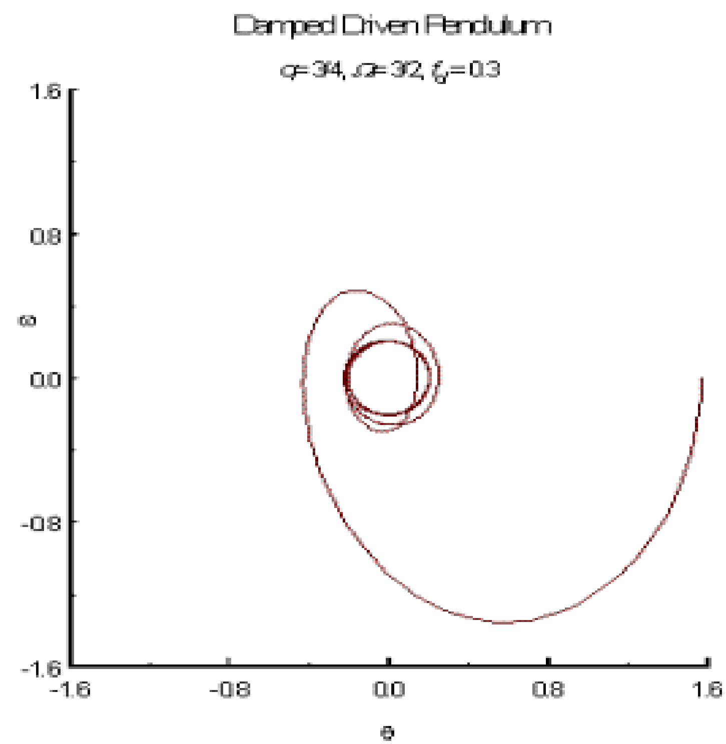
Attractor of Damped (Undriven) Pendulum



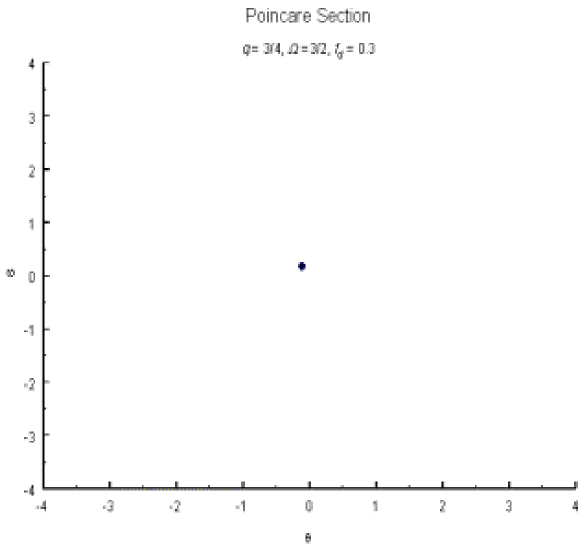


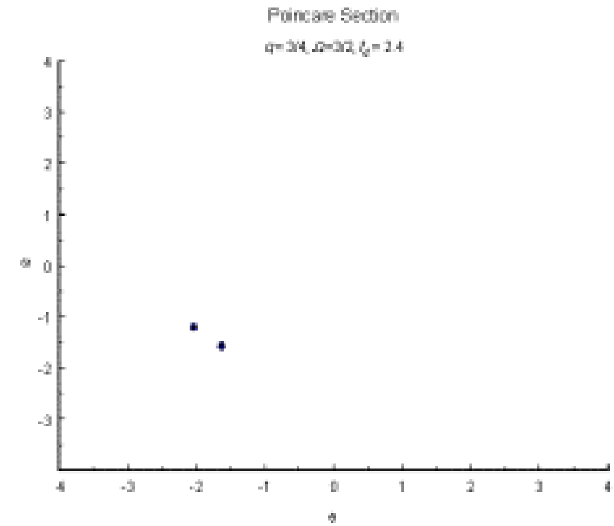
Attractor of Damped Driven Pendulum in the Non-Chaotic Regime

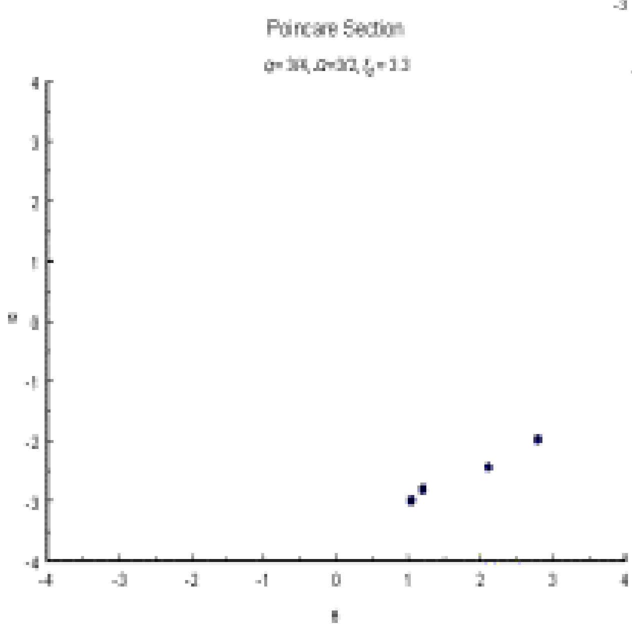




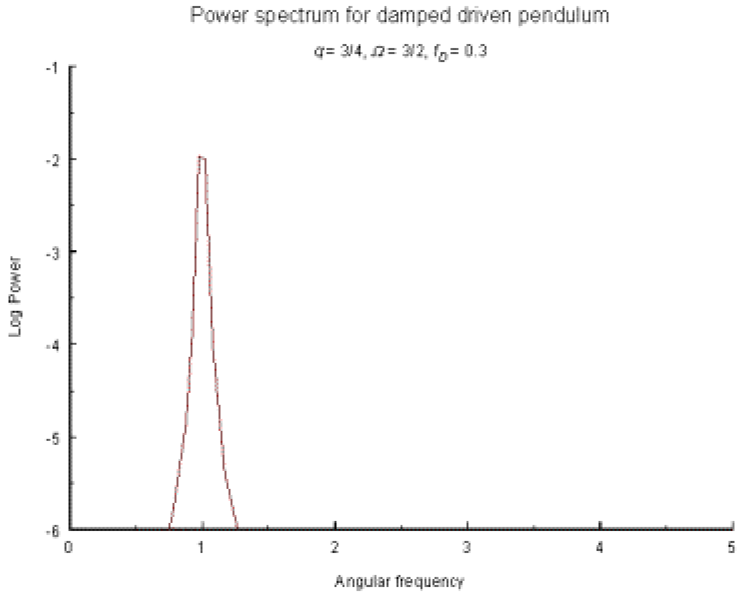
Poincaré Sections Signifying Transition to Transition To Chaos via Period Doubling Route

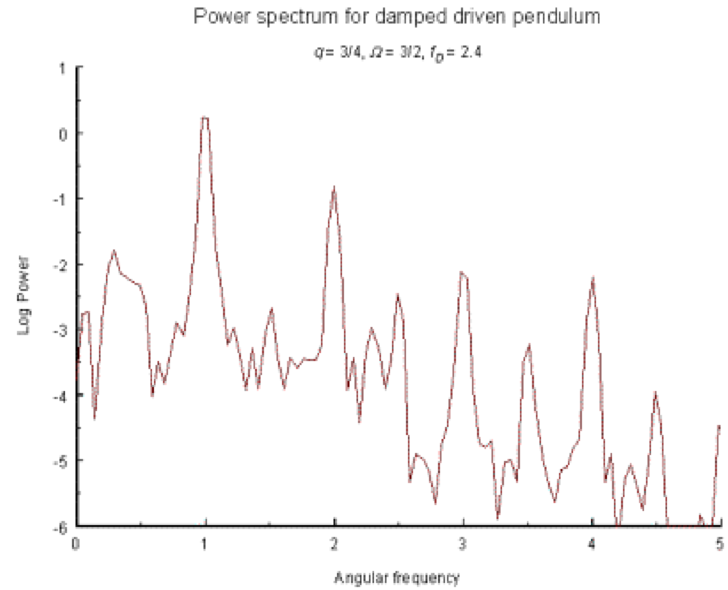




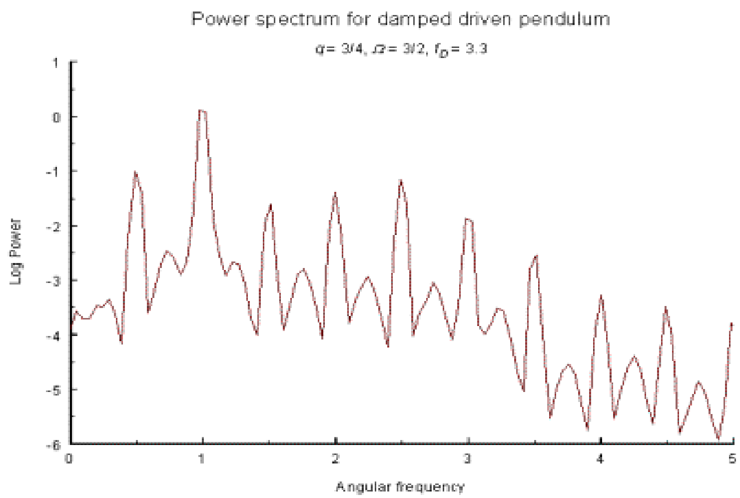


Power Spectrum of in Transition to Chaos



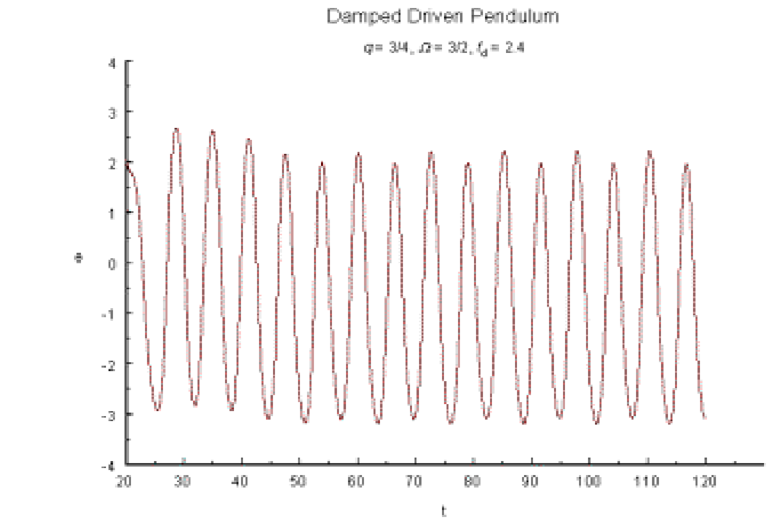


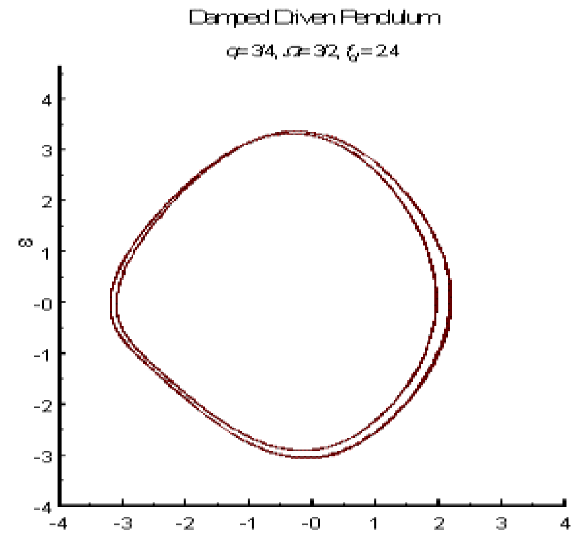
Period-2: Significant power appears at 1/2, 3/2, …, of the driving force frequency

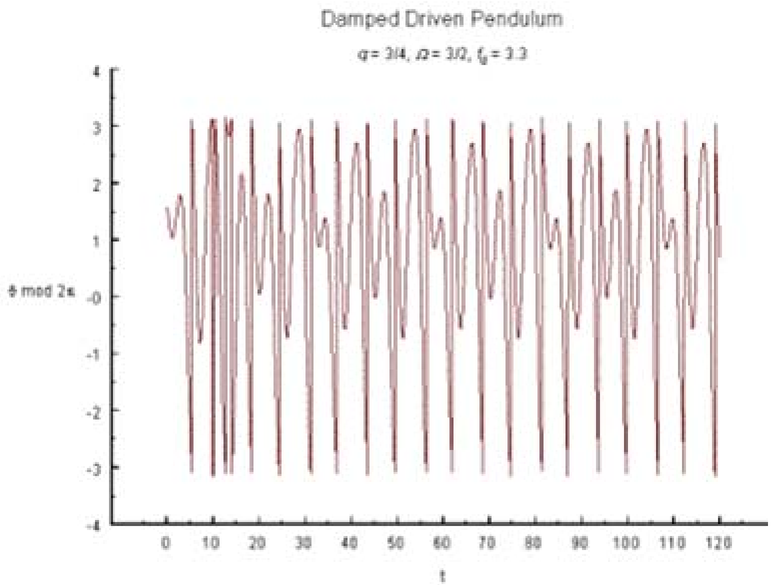


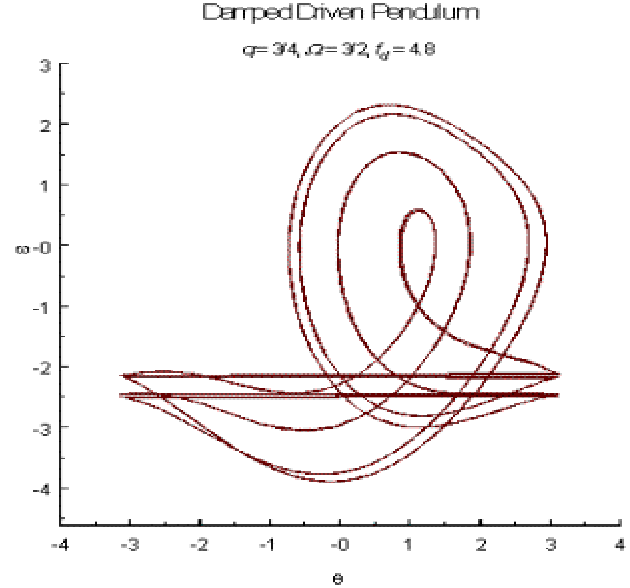
Period-4: There is power at 1/4, 3/4, …, of the driving force frequency

Attractor of Damped Driven Pendulum in Transition to Chaos

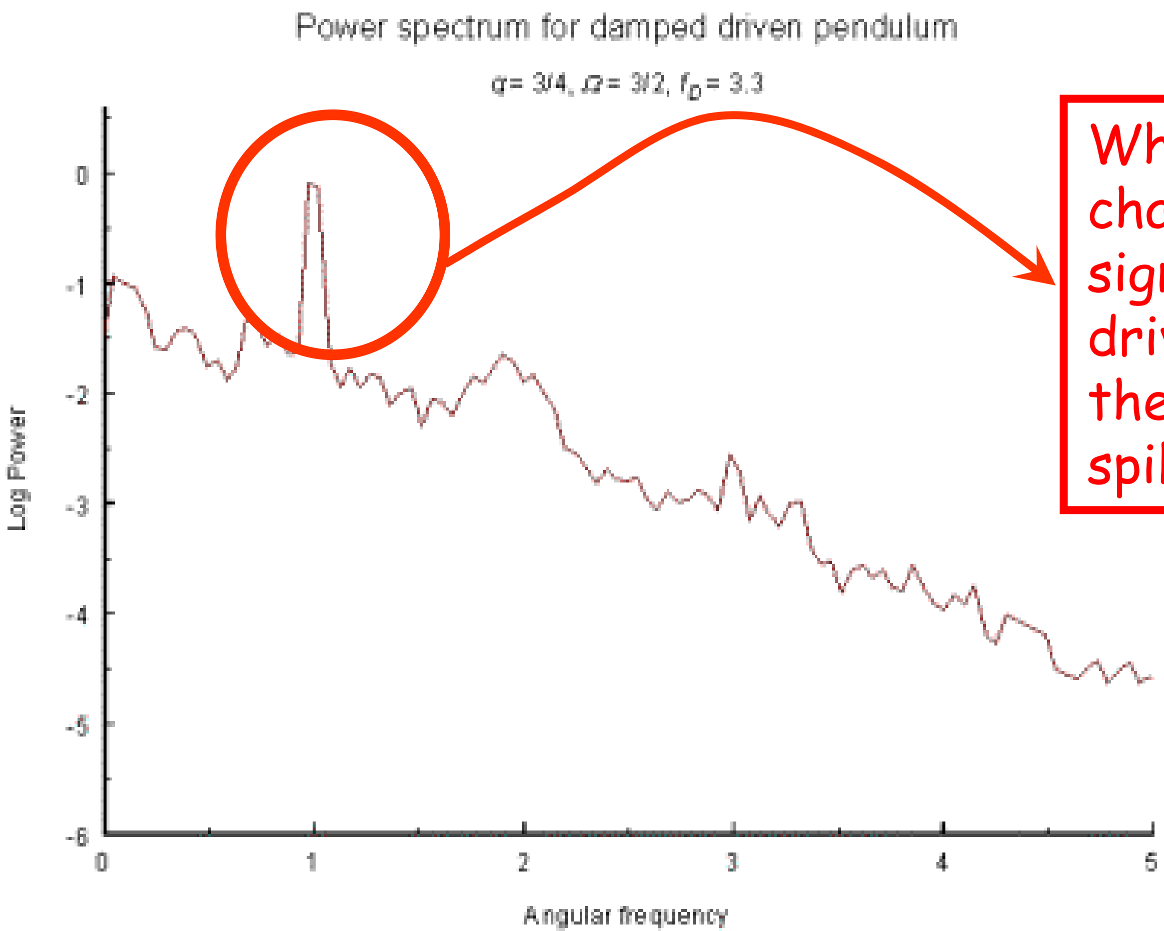






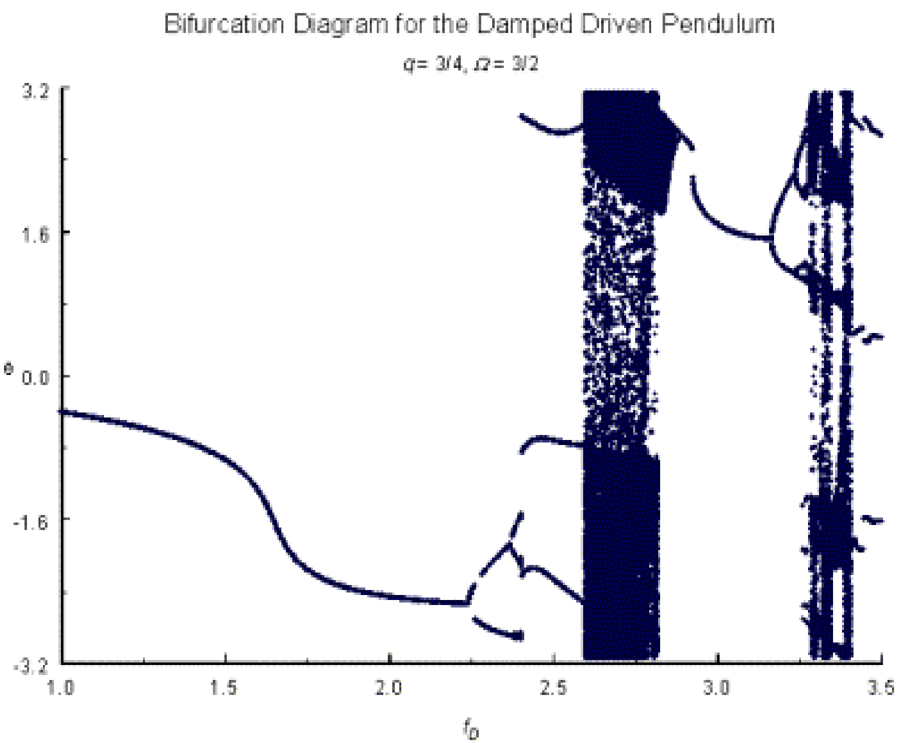


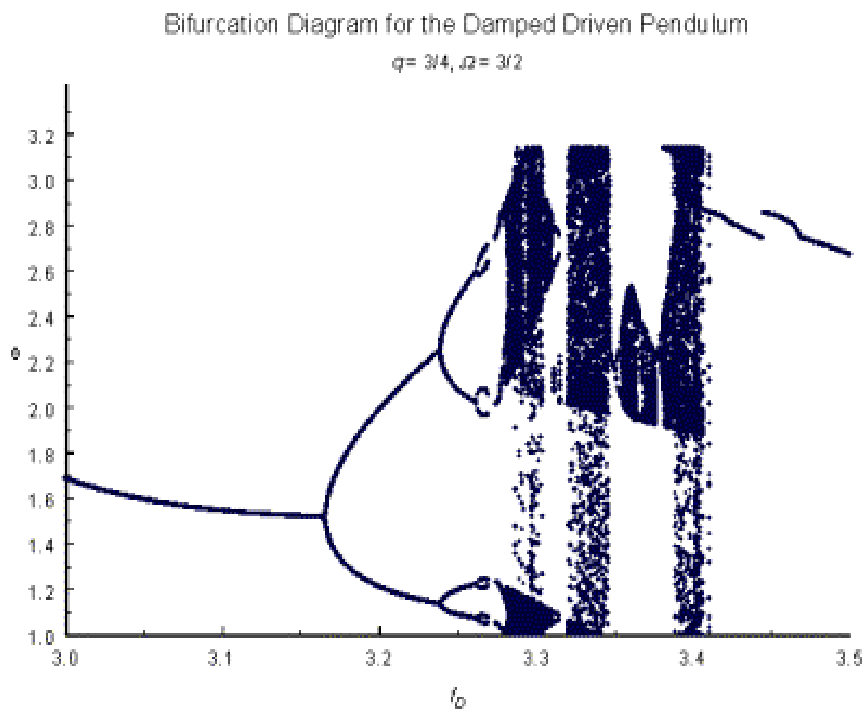
Power Spectrum in the Chaotic Regime



When the system is chaotic, there is still significant power at the drive frequency but there are no other sharp spikes in the spectrum.

Bifurcation Diagram for Damped Driven Pendulum





Feigenbaum Number as an Example of Surprising Order in Chaos

- The ratio of spacings between consecutive values of μ at the bifurcations approaches a universal constant – **the Feigenbaum number**

$$\delta = \lim_{n \to \infty} \frac{F_n - F_{n-1}}{F_{n+1} - F_n} \approx 0.4669$$

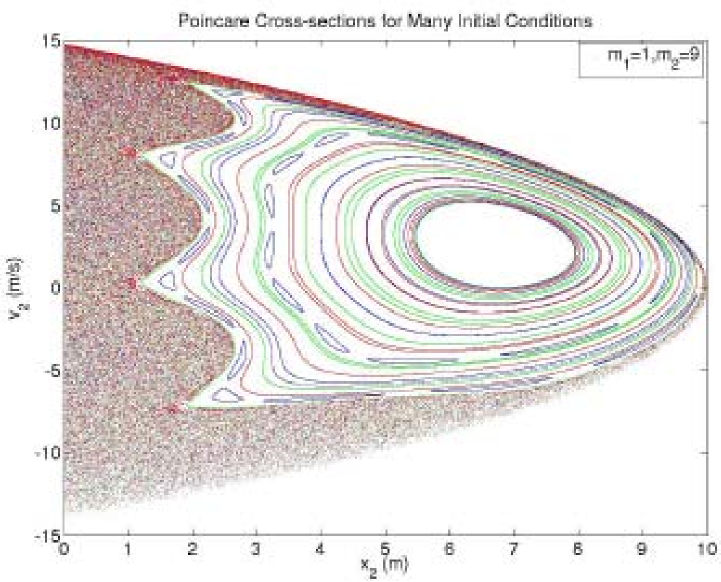
F_n : value at which transition to period-2^n takes place

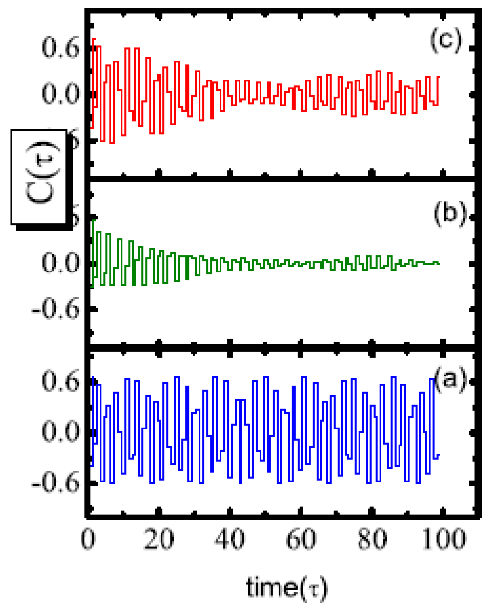
- This is universal to all differential equations (within certain limits) and applies to the pendulum. By using the first few bifurcation points, one can predict the onset of chaos.

Diagnostic Tools for Conservative Chaos in "Two balls in 1D with gravity" Problem

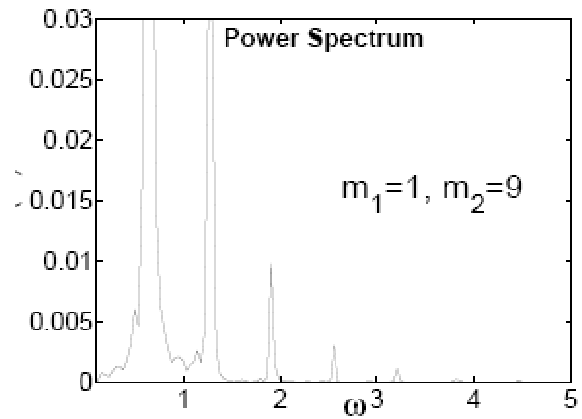
- The dynamical system is chaotic if we find that:

1. Poincare section contains areas which are densely filled with trajectory intersection points
2. Autocorrelation function decays fast to zero
3. Power spectrum displays wide continuum





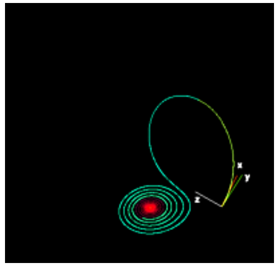
Autocorrelation function for $x_2(t)$ (a) $m_1 = 1.0, m_2 = 1.0$ (b) $m_1 = 1.0, m_2 = 2.0$ (c) $m_1 = 1.0, m_2 = 9.0$

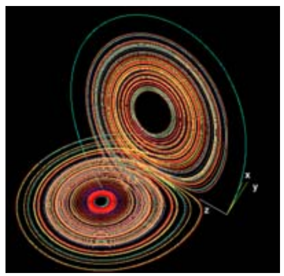


Power spectrum for $m_1 = 1.0, m_2 = 9.0$

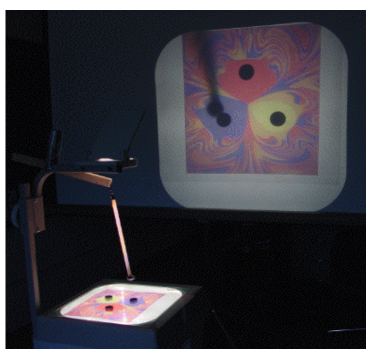
Summary Dissipative vs. Conservative and Permanent vs. Transient Chaos

❑**Permanent:** trajectory in phase space starting from arbitrary initial conditions ends up on strange attractor (Cantor filaments) as a bounded region of phase space where trajectories appear to skip around randomly.

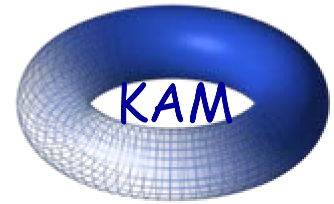


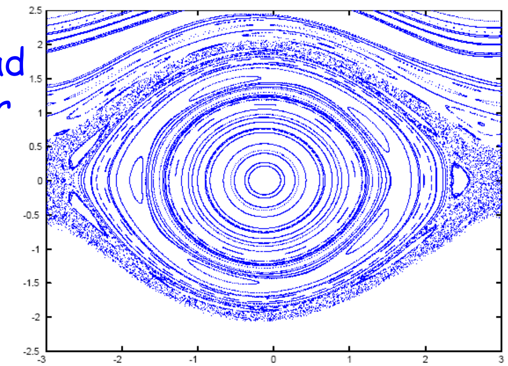


❑**Transient:** typical initial conditions from the fractal basin boundary (Cantor filaments) result in finite time chaotic behavior lasts for finite time while the system is approaching attractors

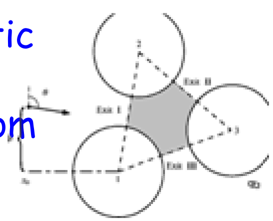


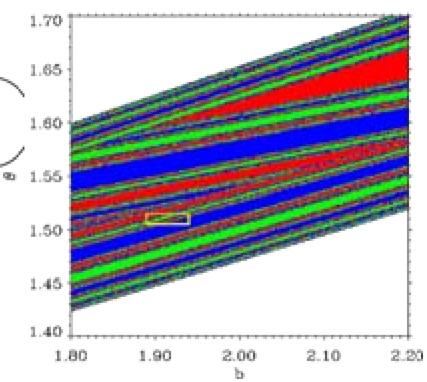
❑**Permanent:** Hierarchically nested pattern of chaotic bands (fat fractals) and regular island – **there are no attractors** (due to conservation of energy and phase space volume) - instead the appearance of regular or chaotic motion strongly depends on the initial conditions and the total energy.





❑**Transient:** Chaotic scattering where motion starting from specific initial conditions (Cantor clouds) is irregular in a finite region of space in which significant forces act.





Do Computers Simulations of Chaos Make Any Sense?

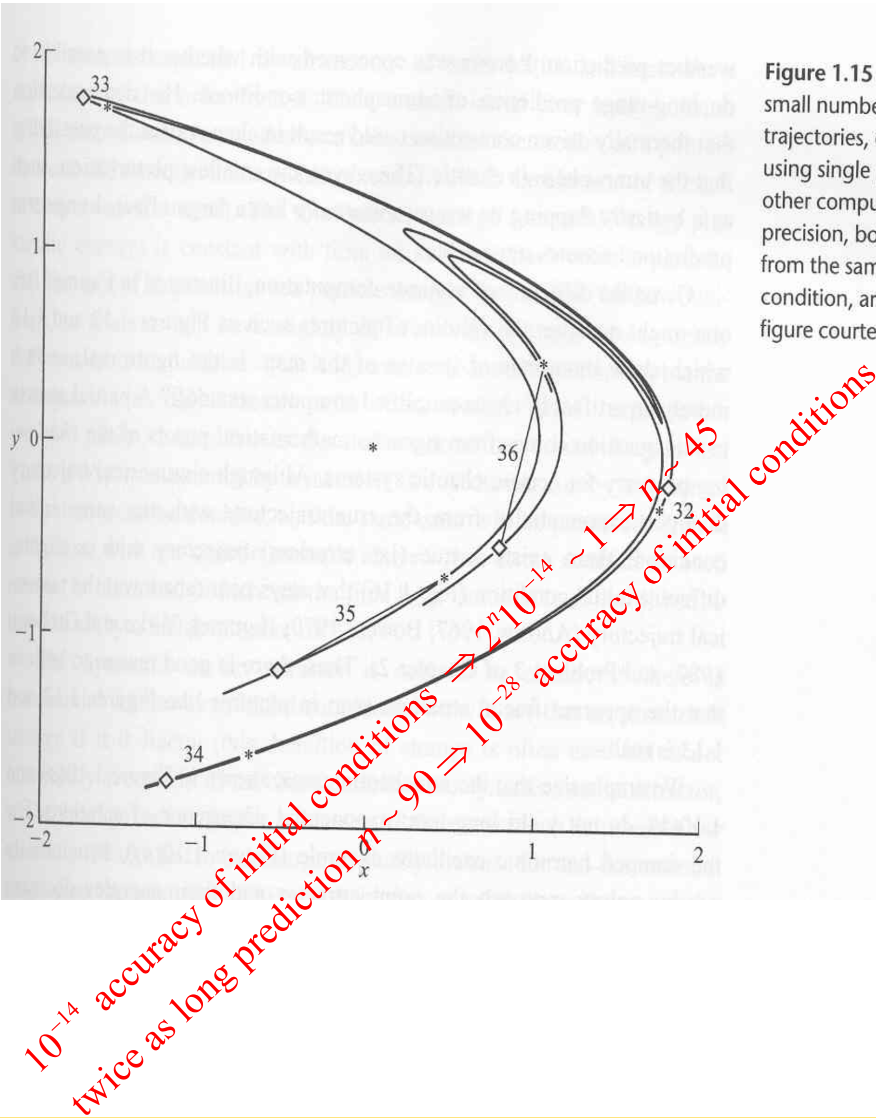


Figure 1.15 After a relatively small number of iterates, two trajectories, one computed using single precision, the other computed using double precision, both originating from the same initial condition, are far apart. (This figure courtesy of Y. Du.)

Shadowing Theorem: Numerically computed chaotic Although a trajectory diverges exponentially from the true trajectory with the same initial coordinates, there exists an errorless trajectory with a slightly different initial condition that stays near ("shadows") the numerically computed one. Therefore, the fractal structure of chaotic trajectories seen in computer maps is real.