LECTURE: Magnons as low-energy excitations in ferro- and antiferro-magnets

1° Exact low energy states of quantum spin chains

$$\hat{H} = J \sum_{i=1}^{N} \hat{\vec{S}}_i \cdot \hat{\vec{S}}_{i+1}$$ ⇒ antiferromagnetic for $J>0$, assume periodic B.C.

$S = \frac{1}{2}$, $\hat{\vec{S}}_i = \hat{I}_1 \otimes \hat{I}_2 \otimes \ldots \otimes \frac{1}{2}\hat{\vec{\sigma}} \otimes \ldots \otimes \hat{I}_N$

↳ exactly solvable for $N \to \infty$ via Bethe ansatz

- eigenstates can be labeled as $|S, S_z, E, q\rangle$ (eigenvalue of $\hat{\vec{S}}^2$; eigenvalue of $\hat{S}_z$; eigenenergy; q-vector)

$$\hat{\vec{S}} = \sum_{i=1}^{N} \hat{\vec{S}}_i$$

a) $N=4$, $q = 0, \pi/2, 3\pi/2 \mod 2\pi$

$$|GS\rangle = \frac{1}{\sqrt{12}}\Big(2|\uparrow\downarrow\uparrow\downarrow\rangle + 2|\downarrow\uparrow\downarrow\uparrow\rangle - |\uparrow\uparrow\downarrow\downarrow\rangle - |\uparrow\downarrow\downarrow\uparrow\rangle - |\downarrow\downarrow\uparrow\uparrow\rangle - |\downarrow\uparrow\uparrow\downarrow\rangle\Big)$$
ground state
$$= |0, 0, -3J, 0\rangle$$

$$|0, 0, -J, \pi\rangle = \frac{1}{\sqrt{4}}\Big(|\downarrow\downarrow\uparrow\uparrow\rangle - |\downarrow\uparrow\uparrow\downarrow\rangle - |\uparrow\downarrow\downarrow\uparrow\rangle + |\uparrow\uparrow\downarrow\downarrow\rangle\Big)$$
↳ excited state

b) $N=6$, $q = 0, \pi/3, 2\pi/3, \pi, 4\pi/3, 5\pi/3 \mod 2\pi$

$$|GS\rangle = \frac{1}{\sqrt{26-6\sqrt{13}}}\Big[\,|\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\rangle - |\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\rangle$$

$$+\frac{1-\sqrt{13}}{6}\Big(|\uparrow\uparrow\downarrow\uparrow\downarrow\downarrow\rangle - |\uparrow\downarrow\uparrow\downarrow\downarrow\uparrow\rangle + |\downarrow\uparrow\downarrow\downarrow\uparrow\uparrow\rangle - |\uparrow\downarrow\downarrow\uparrow\uparrow\downarrow\rangle$$

$$+|\downarrow\downarrow\uparrow\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\downarrow\uparrow\downarrow\rangle - |\downarrow\downarrow\uparrow\downarrow\uparrow\uparrow\rangle + |\downarrow\uparrow\downarrow\uparrow\uparrow\downarrow\rangle$$

$$-|\uparrow\downarrow\uparrow\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\uparrow\downarrow\downarrow\uparrow\rangle - |\uparrow\uparrow\downarrow\downarrow\uparrow\downarrow\rangle + |\uparrow\downarrow\downarrow\uparrow\downarrow\uparrow\rangle\Big)$$

$$+\frac{4-\sqrt{13}}{3}\Big(|\uparrow\uparrow\uparrow\downarrow\downarrow\downarrow\rangle - |\uparrow\uparrow\downarrow\downarrow\downarrow\uparrow\rangle + |\uparrow\downarrow\downarrow\downarrow\uparrow\uparrow\rangle$$

$$-|\downarrow\downarrow\downarrow\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\uparrow\uparrow\uparrow\downarrow\rangle - |\downarrow\uparrow\uparrow\uparrow\downarrow\downarrow\rangle\Big)$$

$$= \left|0, 0, -\frac{J}{2}(5+\sqrt{13}), \pi\right\rangle$$

c) $N=8$

$E_{GS} = -5.65109\,J$ vs. Bethe ansatz $\underset{N\to\infty}{}$ $E_{GS} = -8\ln 2$
$= -5.4518$

d) for N sites $|GS\rangle$ is a linear combination of all $\binom{N}{N/2}$ states with $N/2$ spins $\uparrow$ and $N/2$ spins $\downarrow$

2° Ferromagnetic magnons

- map spin operators to Holstein-Primakoff bosons

$$\hat{S}_i^- = \sqrt{2S}\, \hat{a}_i^+ \sqrt{1 - \frac{\hat{a}_i^+ \hat{a}_i}{2S}}$$

$$\hat{S}_i^+ = \sqrt{2S} \sqrt{1 - \frac{\hat{a}_i^+ \hat{a}_i}{2S}}\, \hat{a}_i$$

$$\hat{S}_i^z = S - \underbrace{\hat{a}_i^+ \hat{a}_i}$$

preserves angular momentum commutation relations

$$[\hat{S}_i^\alpha, \hat{S}_i^\beta] = i \sum_{\gamma=1}^{3} \varepsilon_{\alpha\beta\gamma} \hat{S}_i^\gamma$$

$$[\hat{a}_i, \hat{a}_i^+] = 1$$

↳ eigenvalues $0, 1, 2, \ldots$

↳ eigenvalues $-S, -S+1, \ldots, S-1, S$

$\Rightarrow$ constraint $\hat{a}_i^+ \hat{a}_i \leq 2S$ is "non-holonomic"

$$\hat{S}_i^- |S_z = -S\rangle_i = \sqrt{2S}\, \hat{a}_i^+ \sqrt{1 - \frac{\hat{a}_i^+ \hat{a}_i}{2S}}\, |n = 2S\rangle_i$$

$$= \sqrt{2S}\, \hat{a}_i^+ \sqrt{1 - \frac{2S}{2S}}\, |n=2S\rangle_i \equiv 0$$

$$\hat{S}_i^+ |m = -S-1\rangle_i = \sqrt{2S} \sqrt{1 - \frac{\hat{a}_i^+ \hat{a}_i}{2S}}\, \hat{a}_i |n = 2S+1\rangle_i$$

$$= \sqrt{2S} \sqrt{1 - \frac{\hat{a}_i^+ \hat{a}_i}{2S}}\, \sqrt{2S+1}\, |n = 2S\rangle_i$$

$$= \sqrt{2S} \sqrt{1 - \frac{2S}{2S}}\, \sqrt{2S+1}\, |n = 2S\rangle_i \equiv 0$$

$\rightarrow$ so, $\hat{S}_i^\pm$ do not connect physical subspace $n < 2S$ of an infinite bosonic Hilbert space to the unphysical one $n > 2S+1$

- vacuum of HP bosons satisfies

$$\hat{a}_i^+ \hat{a}_i |0\rangle_i = 0 \Rightarrow \hat{S}_i^z |0\rangle_i = S|0_i\rangle$$

So $|0\rangle_i$ is different representation of fully polorized ferromagnetic ground state $|\uparrow\uparrow \dots \uparrow\rangle$

- in practice $\sqrt{1 - \frac{\hat{a}_i^+\hat{a}_i}{2S}}$ is never used,

and instead it is expounded in orders of $\hat{a}_i^+\hat{a}_i / 2S$

$$\hat{H} = -\underset{0}{J} \sum_{\langle ij \rangle} \hat{\vec{S}}_i \cdot \hat{\vec{S}}_j$$

$\hookrightarrow$ only nearest-neighbors (NN) interact

$$= -J \sum_{\langle ij \rangle} \left(\frac{\hat{S}_i^+ \hat{S}_j^- + \hat{S}_i^- \hat{S}_j^+}{2} + \hat{S}_i^z \hat{S}_j^z \right)$$

$$= -J \sum_{\langle ij \rangle} \left[S \sqrt{1 - \frac{\hat{a}_i^+\hat{a}_i}{2S}}\, \hat{a}_i \hat{a}_j^+ \sqrt{1 - \frac{\hat{a}_j^+\hat{a}_j}{2S}} \right.$$

$$\left. + S \hat{a}_i^+ \sqrt{1 - \frac{\hat{a}_i^+\hat{a}_i}{2S}} \sqrt{1 - \frac{\hat{a}_j^+\hat{a}_j}{2S}}\, \hat{a}_j + (S - \hat{a}_i^+\hat{a}_i)(S - \hat{a}_j^+\hat{a}_j) \right]$$

no. of NN

$$= \underbrace{-N z \frac{JS^2}{2}}_{\text{order } (1/S)^{-2}} + \underbrace{JS \sum_{\langle ij \rangle} \left(\hat{a}_i^+\hat{a}_i + \hat{a}_j^+\hat{a}_j - \hat{a}_i^+\hat{a}_j - \hat{a}_j^+\hat{a}_i \right)}_{\text{order} (1/S)^{-1}}$$

$$\underbrace{-J \sum_{\langle ij \rangle} \left[\hat{a}_i^+\hat{a}_i \hat{a}_j^+\hat{a}_j - \frac{1}{4}\left(\hat{a}_i^+\hat{a}_i^+\hat{a}_i\hat{a}_j + \hat{a}_i^+\hat{a}_j^+\hat{a}_j\hat{a}_j + \hat{a}_j^+\hat{a}_i^+\hat{a}_i\hat{a}_i + \hat{a}_j^+\hat{a}_j^+\hat{a}_j\hat{a}_i \right) \right]}_{\text{order } (1/S)^0} + \mathcal{O}(1/S)$$

→ in the classical limit $S \to \infty$, we keep only first two terms (keeping only first is too crude approximation as it gives constant energy)

→ second term is diagonalized by a Fourier tranform:

$$\boxed{\hat{a}_i = \frac{1}{\sqrt{N}} \sum_{\vec{q}} e^{i\vec{q}\cdot\vec{R}} \hat{a}_{\vec{q}} \qquad 1/N \sum_{\vec{q}} e^{i\vec{q}(\vec{R}_i - \vec{R}_j)} = \delta_{ij} \qquad 1/N \sum_{i} e^{i(\vec{q}-\vec{q}')\vec{R}_i} = \delta_{\vec{q}\vec{q}'}}$$

$$\hat{H}_{1/S} = -Nz\frac{JS^2}{2} + \frac{JS}{N} \sum_{\vec{q}\vec{q}'} \sum_{\langle ij \rangle} \left(e^{-i\vec{q}\vec{R}_i + i\vec{q}'\vec{R}_i} + e^{-i\vec{q}\vec{R}_j + i\vec{q}'\vec{R}_j} - e^{-i\vec{q}\vec{R}_i + i\vec{q}'\vec{R}_j} - e^{-i\vec{q}\vec{R}_j + i\vec{q}'\vec{R}_i} \right) \hat{a}^+_{\vec{q}} \hat{a}_{\vec{q}'}$$

$$= -Nz\frac{JS^2}{2} + \frac{JS}{2N} \sum_{\vec{q}\vec{q}'} \sum_{i} \sum_{\Delta\vec{R}} e^{-i(\vec{q}-\vec{q}')\cdot\vec{R}_i} \left(1 + e^{-i\vec{q}\cdot\Delta\vec{R} + i\vec{q}'\Delta\vec{R}} - e^{-i\vec{q}'\cdot\Delta\vec{R}} - e^{-i\vec{q}\cdot\Delta\vec{R}}\right) \hat{a}^+_{\vec{q}} \hat{a}_{\vec{q}'}$$

$$= -Nz\frac{JS^2}{2} + \frac{JS}{\cancel{2}} \sum_{\vec{q}} \sum_{\Delta\vec{R}} \left(\cancel{2} - \cancel{2}\cos \vec{q}\cdot\Delta\vec{R}\right) \hat{a}^+_{\vec{q}} \hat{a}_{\vec{q}}$$

$$= -Nz\frac{JS^2}{2} + \sum_{\vec{q}} JS \sum_{\Delta\vec{R}} \left(1 - \cos \vec{q}\cdot\Delta\vec{R}\right) \hat{a}^+_{\vec{q}} \hat{a}_{\vec{q}}$$

↳ sum over all NN vectors $\Delta\vec{R}$

simple cubic lattice →

$$= \underbrace{-\frac{NJS^2}{3}}_{\text{const.}} + \sum_{\vec{q}} \underbrace{2JS(3 - \cos q_x a - \cos q_y a - \cos q_z a)}_{\hbar\omega_{\vec{q}}} \hat{a}^+_{\vec{q}} \hat{a}_{\vec{q}}$$

magnon energy-momentum dispersion $\hbar\omega_{\vec{q}}$

$\Rightarrow$ using:

$$\frac{d\hat{S}^-_{\vec{q}}}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{S}^-_{\vec{q}}], \quad \hat{S}^-_{\vec{q}} = \sqrt{2S}\,\hat{a}^+_{\vec{q}}\sqrt{1 - \frac{\hat{a}^+_{\vec{q}}\hat{a}_{\vec{q}}}{2S}}$$

$$\hbar\omega_{\vec{q}}\,\hat{S}^-_{\vec{q}}|GS\rangle = [\hat{H}, \hat{S}^-_{\vec{q}}]|GS\rangle$$

$$= \hat{H}\hat{S}^-_{\vec{q}}|GS\rangle - \hat{S}^-_{\vec{q}} E_{GS}|GS\rangle$$

$$\hat{H}\hat{S}^-_{\vec{q}}|GS\rangle = (E_{GS} + \hbar\omega_{\vec{q}})\hat{S}^-_{\vec{q}}|GS\rangle$$

$\Rightarrow$ we find that $\hat{S}^-_{\vec{q}}|GS\rangle$ is eigenstate of $\hat{H}$ with eigenenergy $E_{GS} + \hbar\omega_{\vec{q}}$, so $\hat{S}^-_{\vec{q}}$ creates one quantum of energy or Fock state of one magnon:

$$|1_{\vec{q}}\rangle = \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} e^{iqx_n}|\underbrace{\uparrow\cdots\uparrow}_{n\ \text{times}}\downarrow\underbrace{\uparrow\cdots\uparrow}_{N-n-1\ \text{times}}\rangle$$

for 1D chain of $S=1/2$ in firs-quantized notation

- $\hat{S}^z_i(\hat{S}^-_{\vec{q}}|GS\rangle) = \hbar\left(S - \frac{1}{N}\right)(\hat{S}^-_{\vec{q}}|GS\rangle) \Rightarrow \underbrace{\hat{S}^z}_{\sum_{i=1}^{N}\hat{S}^z_i}(\hat{S}^-_{\vec{q}}|GS\rangle) = \hbar(SN-1)(\hat{S}^-_{\vec{q}}|GS\rangle)$
- $\hbar\omega_{\vec{q}} = 0 \Rightarrow \vec{q} = 0$ excitation, as uniform macrospin rotation, does not cost any energy which is a special case of Goldstone theorem $\rightarrow$ spontaneous breaking of a continuous symmetry (here spin rotation) is always accompanied by zero energy mode as Goldstone boson

naive $\hat{S}^-_{\vec{q}}\hat{S}^-_{\vec{q}'}|GS\rangle$ is no longer eigenstate of $\hat{H}$, but it is eigenstate of $\hat{H}_{I/S}$ when magnons do not interact

■ magnetization at finite temperature in approximation of noninteracting magnons

$$\langle \hat{a}^{+}_{\vec{q}}\, a_{\vec{q}} \rangle_T = \langle \hat{n}_{\vec{q}} \rangle_T = \underbrace{\frac{1}{e^{\beta\hbar\omega_{\vec{q}}}-1}}_{\text{Bose-Einstein with } \mu=0}$$

↳ quantum-statistical average $\langle \dots \rangle = \mathrm{Tr}(\hat{\rho}_{eq} \dots)$

$\beta = 1/k_B T$

$$\langle \hat{n} \rangle_T = \sum_{\vec{q}} \langle \hat{n}_{\vec{q}} \rangle = V_d \int_{BZ} \frac{d^d q}{(2\pi)^3} \frac{1}{e^{\beta\hbar\omega_{\vec{q}}}-1}$$

↑ total number of excited magnons at T

$$M(T) = \frac{g\mu_B (NS - \langle \hat{n} \rangle)}{V} = \underbrace{\frac{g\mu_B NS}{V}}_{M_{sat}} - \underbrace{g\mu_B \int_{BZ} \frac{d^d q}{(2\pi)^3} \frac{1}{e^{\beta\hbar\omega_{\vec{q}}}-1}}_{\text{dominated by } \vec{q}\to 0\text{, so:}}$$

- $\hbar\omega_{\vec{q}} \approx 2JS\,a^2 q^2$
- replace $\int_{BZ}$ with $\int_{\mathbb{R}^3}$

$$d=3 \Rightarrow M \approx M_{sat} - g\mu_B \int_{\mathbb{R}^3} \frac{d^3 q}{(2\pi)^3} \frac{1}{e^{\beta JSa^2q^2}-1}$$

$$= M_{sat} - g\mu_B \frac{\zeta(3/2)}{8} \frac{1}{(\pi/\beta JSa^2)^{3/2}}$$

$T^{3/2}$ Bloch law

$d = 1, 2$:

$$\int_\lambda dq\, q^{d-3} \sim \begin{cases} 1/\lambda, & d=1 \\ \ln \lambda, & d=2 \end{cases}$$

which is divergent as $\lambda \to 0 \Rightarrow$ Mermin-Wagner theorem

3° Antiferromagnetic magnons

$i \in$ sublattice A, $j \in$ sublattice B

$$\hat{S}_i^- = \sqrt{2S}\,\hat{a}_i^+ \sqrt{1 - \frac{\hat{a}_i^+ \hat{a}_i}{2S}} \qquad \hat{S}_j^- = \sqrt{2S}\sqrt{1 - \frac{\hat{b}_j^+ \hat{b}_j}{2S}}\,\hat{b}_j$$

$$\hat{S}_i^+ = \sqrt{2S}\sqrt{1 - \frac{\hat{a}_i^+ \hat{a}_i}{2S}}\,\hat{a}_i \qquad \hat{S}_j^+ = \sqrt{2S}\,\hat{b}_j^+ \sqrt{1 - \frac{\hat{b}_j^+ \hat{b}_j}{2S}}$$

$$\hat{S}_i^z = S - \hat{a}_i^+ \hat{a}_i \qquad \hat{S}_j^z = -S + \hat{b}_j^+ b_j$$

bosonization based on spin-flips on top of Néel state, which is not correct GS, but this will improve it eventually

$$\rightarrow |0\rangle = \prod_{i \in A}^{\otimes} |S\rangle_i \prod_{j \in B}^{\otimes} |-S\rangle_j \,, \quad \hat{a}_i|0\rangle = \hat{b}_j|0\rangle = 0$$

$$\hat{H} = -\underset{<0}{J} \sum_{\langle ij \rangle} \hat{\vec{S}}_i \cdot \hat{\vec{S}}_j$$

$$= N z \frac{JS^2}{2} - JS \sum_{\langle ij \rangle} \left(\hat{a}_i^+ \hat{a}_i + \hat{b}_j^+ \hat{b}_j + \hat{a}_i \hat{b}_j + \hat{b}_j^+ \hat{a}_i^+ \right)$$

+ interaction terms with 4, 6, ... bosonic operators

$\rightarrow$ keep only qudratic terms and diagonalize with:

$$\hat{a}_i = \sqrt{\frac{2}{N}} \sum_{\vec{q}} e^{i\vec{q}\cdot\vec{R}_i}\,\hat{a}_{\vec{q}}$$

$$\hat{b}_j = \sqrt{\frac{2}{N}} \sum_{\vec{q}} e^{-i\vec{q}\cdot\vec{R}_j}\,\hat{b}_{\vec{q}}$$

1/N half

opposite sign

$$\hat{H} = Nz\,\frac{JS^2}{2} - \frac{2JS}{N}\sum_{\vec{q}\,\vec{q}\,'}\sum_{i\in A}\sum_{\Delta\vec{R}}\Big[e^{-i(\vec{q}-\vec{q}\,')\cdot\vec{R}_i}\,\hat{a}^+_{\vec{q}}\,\hat{a}_{\vec{q}\,'}$$
$$+ e^{i(\vec{q}-\vec{q}\,')\cdot(\vec{R}_i+\Delta\vec{R})}\,\hat{b}^+_{\vec{q}}\,\hat{b}_{\vec{q}\,'}$$
$$+ e^{i(\vec{q}-\vec{q}\,')\cdot\vec{R}_i - i\vec{q}\,'\cdot\Delta\vec{R}}\,\hat{a}_{\vec{q}}\,\hat{b}_{\vec{q}\,'}$$
$$+ e^{i(\vec{q}-\vec{q}\,')\cdot\vec{R}_i + i\vec{q}\cdot\Delta\vec{R}}\,\hat{b}^+_{\vec{q}}\,\hat{a}^+_{\vec{q}\,'}\Big]$$
$$= Nz\,\frac{JS^2}{2} - JS\sum_{\vec{q}}\sum_{\Delta\vec{R}}\Big(\hat{a}^+_{\vec{q}}\hat{a}_{\vec{q}} + \hat{b}^+_{\vec{q}}\hat{b}_{\vec{q}} + e^{-i\vec{q}\cdot\Delta\vec{R}}\,\hat{a}_{\vec{q}}\,\hat{b}_{\vec{q}} + e^{i\vec{q}\cdot\Delta\vec{R}}\,\hat{b}^+_{\vec{q}}\,\hat{a}^+_{\vec{q}}\Big)$$

$\rightarrow$ for simple cubic lattice in d-dimensions:

$$\hat{H} = Nz\,\frac{JS^2}{2} - JS\sum_{\vec{q}}\Big(z\,\hat{a}^+_{\vec{q}}\hat{a}_{\vec{q}} + z\,\hat{b}^+_{\vec{q}}\hat{b}_{\vec{q}} + 2\sum_{\nu=1}^{d}\cos(q_\nu a)\,\hat{a}_{\vec{q}}\,\hat{b}_{\vec{q}} + 2\sum_{\nu=1}^{d}\cos(q_\nu a)\,\hat{b}^+_{\vec{q}}\,\hat{a}^+_{\vec{q}}\Big)$$

$$z = 2d\,, \qquad \gamma_{\vec{q}} = \frac{1}{d}\sum_{\nu=1}^{d}\cos(q_\nu a)$$

$$\hat{H} = Nz\,\frac{JS^2}{2} - zJS\sum_{\vec{q}}\Big(\hat{a}^+_{\vec{q}}\hat{a}_{\vec{q}} + \hat{b}^+_{\vec{q}}\hat{b}_{\vec{q}} + \gamma_{\vec{q}}\,a_{\vec{q}}\,b_{\vec{q}} + \gamma_{\vec{q}}\,\hat{a}^+_{\vec{q}}\,b^+_{\vec{q}}\Big)$$

$\rightarrow$ third and forth term do not conserve the total number of bosons $\Rightarrow$ diagonalize $\hat{H}$ with additional Bogoliubov-Valatin transformation

$$\hat{a}_{\vec{q}} = \cosh\theta_{\vec{q}}\,\hat{\alpha}_{\vec{q}} - \sinh\theta_{\vec{q}}\,\hat{\beta}^{+}_{\vec{q}}$$

$$\hat{b}_{\vec{q}} = -\sinh\theta_{\vec{q}}\,\hat{\alpha}^{+}_{\vec{q}} + \cosh\theta_{\vec{q}}\,\hat{\beta}_{\vec{q}}$$

$$\hat{\alpha}_{q}\hat{\beta}_{q} = 0 = \hat{\beta}^{+}_{\vec{q}}\hat{\alpha}^{+}_{\vec{q}} \quad \text{if } \tanh 2\theta_{\vec{q}} = \gamma_{\vec{q}}$$

$$\hat{H} = Nz\frac{JS^2}{2} - zJS\sum_{\vec{q}}\left(\cosh^2\theta_{\vec{q}} + \sinh^2\theta_{\vec{q}} - 2\gamma_{\vec{q}}\cosh\theta_{\vec{q}}\sinh\theta_{\vec{q}}\right)\left(\hat{\alpha}^{+}_{\vec{q}}\hat{\alpha}_{\vec{q}} + \hat{\beta}^{+}_{\vec{q}}\hat{\beta}_{\vec{q}}\right)$$

$$- 2JS\sum_{\vec{q}}\left(2\sinh^2\theta_{\vec{q}} - 2\gamma_{\vec{q}}\cosh\theta_{\vec{q}}\sinh\theta_{\vec{q}}\right)$$

arises from $[\hat{\alpha}_{\vec{q}}, \hat{\alpha}^{+}_{\vec{q}}] = 1 = [\hat{\beta}_{\vec{q}}, \hat{\beta}^{+}_{\vec{q}}]$

$$= Nz\frac{JS^2}{2} - zJS\sum_{\vec{q}}\sqrt{1-\gamma^2_{\vec{q}}}\left(\hat{\alpha}^{+}_{\vec{q}}\hat{\alpha}_{\vec{q}} + \hat{\beta}^{+}_{\vec{q}}\hat{\beta}_{\vec{q}}\right)$$

$$- zJS\sum_{\vec{q}}\left(\sqrt{1-\gamma^2_{\vec{q}}} - 1\right)$$

$$= Nz\frac{JS(S+1)}{2} - zJS\sum_{\vec{q}}\sqrt{1-\gamma^2_{\vec{q}}}\left(\hat{\alpha}^{+}_{\vec{q}}\hat{\alpha}_{\vec{q}} + \frac{1}{2} + \hat{\beta}^{+}_{\vec{q}}\hat{\beta}_{\vec{q}} + \frac{1}{2}\right)$$

N/2 terms in this sum

$$= Nz\frac{JS(S+1)}{2} + \sum_{\vec{q}}\hbar\omega_{\vec{q}}\left(\hat{\alpha}^{+}_{\vec{q}}\hat{\alpha}_{\vec{q}} + \frac{1}{2} + \hat{\beta}^{+}_{\vec{q}}\hat{\beta}_{\vec{q}} + \frac{1}{2}\right)$$

"zero point energy" or "quantum spin fluctuations"

but really second-quantized description of entangled GS

$$\hat{\alpha}_{\vec{q}}\,|GS\rangle_{NIM} = \hat{\beta}_{\vec{q}}\,|GS\rangle_{NIM} = 0$$

↳ non-interacting magnons

$$E_{GS}^{NIM} = Nz\,\frac{JS(S+1)}{2} + \sum_{\vec{q}} \hbar\omega_{\vec{q}}$$

$$= Nz\,\frac{JS(S+1)}{2} - zJS\sum_{\vec{q}}\sqrt{1-\gamma_{\vec{q}}^2}$$

$$= Nz\,\frac{JS(S+1)}{2} - zJS\,\frac{Na^d}{2}\int\frac{d^dq}{(2\pi)^d}\sqrt{1-\left(\frac{1}{d}\sum_{\nu=1}^{d}\cos q_\nu a\right)^2}$$

$$= dNJS^2\cdot\begin{cases}(1+0.363/S), & d=1\\ (1+0.158/S), & d=2\\ (1+0.097/S), & d=3\end{cases}$$

→ compare with: $E_{Néel} = -J\sum_{\langle ij\rangle}\langle Néel|\overbrace{\hat{\vec{S}}_i\cdot\hat{\vec{S}}_j}^{-S^2}|Néel\rangle$

$$= JS^2\,\frac{N}{2}\,z = dNJS^2$$

$$\boxed{\begin{array}{l} E_{GS}^{NIM} < E_{Néel} \\ \text{as } J<0 \end{array}}$$

■ sublattice magnetization per site: $M = \langle \hat{S}_i^z\rangle_{i\in A} = \langle \hat{S}_j^z\rangle_{j\in B}$

$$M_{GS}^{NIM} = {}_{NIM}\langle GS|\hat{S}_i^z|GS\rangle_{NIM}$$

$$= {}_{NIM}\langle GS|S-\hat{a}_i^\dagger\hat{a}_i|GS\rangle_{NIM} = S-\frac{2}{N}\sum_{\vec{q}}{}_{NIM}\langle GS|\hat{a}_{\vec{q}}^\dagger\hat{a}_{\vec{q}}|GS\rangle_{NIM}$$

$$= S-\frac{2}{N}\sum_{\vec{q}}{}_{NIM}\langle GS|\cosh^2\theta_{\vec{q}}\,\hat{\alpha}_{\vec{q}}^\dagger\hat{\alpha}_{\vec{q}} + \sinh^2\theta_{\vec{q}}\,\hat{\beta}_{\vec{q}}^\dagger\hat{\beta}_{\vec{q}}$$

$$-\cosh\theta_{\vec{q}}\sinh\theta_{\vec{q}}\,\hat{\alpha}_{\vec{q}}\hat{\beta}_{\vec{q}} - \cosh\theta_{\vec{q}}\sinh\theta_{\vec{q}}\,\hat{\beta}_{\vec{q}}^\dagger\hat{\alpha}_{\vec{q}}^\dagger|GS\rangle_{NIM}$$

$$M_{GS}^{NIM} = S - \frac{2}{N}\sum_{\vec{q}}' \sinh^2\theta_{\vec{q}} = S - \frac{2}{N}\sum_{\vec{q}}\left(\frac{1}{2}\frac{1}{\sqrt{1-\gamma_{\vec{q}}^2}} - \frac{1}{2}\right)$$

$$= S + \frac{1}{2} - \frac{1}{2}a^d \int \frac{d^d q}{(2\pi)^d}\frac{1}{\sqrt{1-\gamma_{\vec{q}}^2}}$$

$$d=1 \Rightarrow \int dq \frac{1}{\sqrt{1-(1-q^2a^2/2)^2}} \approx \int dq \frac{1}{q^2a^2} = \frac{1}{a}\underbrace{\int\frac{dq}{q}}_{\ln q}$$

↳ diverges logarithmically for $q \to 0$
so in $d=1$ even GS, at $T=0$, does not show long-range order

→ in $d=1$ ferromagnet, long-range order is destroyed at $T>0$ by thermal fluctuations, so for $d=1$ antiferromagnet, with no order at $T=0$, it is common to use jargon of "quantum fluctuations" destroying long-range order

→ for $d=2,3$:

$$M_{GS}^{NIM} = \begin{cases} S(1-0{,}197/S), & d=2 \\ S(1-0{,}078/S), & d=3 \end{cases}$$

So, for $d=2$ and $S=1/2$, M_{GS}^{NIM} is about 40% smaller from $M_{\text{Néel}}$

■ energy momentum dispersion of magnons:

$$\hbar\omega_{\vec{q}} = -zJS\sqrt{1-\gamma_{\vec{q}}^2} = \underbrace{-zJS}_{V_0}\sqrt{1-\left(\frac{1}{d}\sum_{\nu=1}^{d}\cos q_\nu a\right)^2}$$

$\rightsquigarrow$ for $q \to 0$:

$$\hbar\omega_{\vec{q}} \approx zJS\sqrt{1-\left(1-\frac{1}{2d}q^2a^2\right)^2} \approx -zJS\sqrt{\frac{1}{d}q^2a^2}$$

$$= -\frac{zJS}{\sqrt{d}}\,qa = \underbrace{-2\sqrt{d}\,JSqa}_{\text{linear in } q=|\vec{q}|}$$

$\hbar\omega_{\vec{q}=0} \equiv 0$ satisfies Goldstone theorem

■ staggered magnetization at $T>0$ in $d=3$:

$$M(T) = \langle \hat{S}_i^z \rangle_T = \langle S - \hat{a}_i^+ \hat{a}_i \rangle_T = S - \frac{2}{N}\sum_{\vec{q}} \langle \hat{a}_{\vec{q}}^+ \hat{a}_{\vec{q}} \rangle$$

$$= S - \frac{2}{N}\sum_{\vec{q}} \Big\langle \cosh^2\theta_{\vec{q}}\, \hat{\alpha}_{\vec{q}}^+ \hat{\alpha}_{\vec{q}} + \sinh^2\theta_{\vec{q}}\, \hat{\beta}_{\vec{q}}^+ \hat{\beta}_{\vec{q}}$$

$$- \cosh\theta_{\vec{q}} \sinh\theta_{\vec{q}}\, \hat{\alpha}_{\vec{q}} \hat{\beta}_{\vec{q}} - \cosh\theta_{\vec{q}} \sinh\theta_{\vec{q}}\, \hat{\beta}_{\vec{q}}^+ \hat{\alpha}_{\vec{q}}^+ \Big\rangle_T$$

$$= S - \frac{2}{N}\sum_{\vec{q}} \left[\cosh^2\theta_{\vec{q}}\, n_{BE}(\hbar\omega_{\vec{q}}) + \sinh^2\theta_{\vec{q}} \left(n_{BE}(\hbar\omega_{\vec{q}}) + 1\right)\right]$$

$$= \boxed{M_{GS}^{NIM}} - \frac{2}{N}\sum_{\vec{q}} \left(\cosh^2\theta_{\vec{q}} + \sinh^2\theta_{\vec{q}}\right) n_{BE}(\hbar\omega_{\vec{q}})$$

$$\xrightarrow{\text{recall}} S - \frac{2}{N}\sum_{\vec{q}} \sinh^2\theta_{\vec{q}}$$

$$= M_{GS}^{NIM} - \frac{2}{N}\sum_{\vec{q}} \frac{1}{\sqrt{1-\gamma_{\vec{q}}^2}}\,\frac{1}{e^{\beta\hbar\omega_{\vec{q}}}-1}$$

$$= M_{GS}^{NIM} - a^3\int \frac{d^3q}{(2\pi)^3}\,\frac{1}{\sqrt{1-\gamma_{\vec{q}}^2}}\,\frac{1}{e^{\beta\hbar\omega_{\vec{q}}}-1}$$

$$M(T) \approx M_{GS}^{NIM} - \frac{a^3}{2\pi^2} \int_0^\infty dq\, q^2 \frac{\sqrt{3}}{qa} \frac{1}{e^{-2\sqrt{3}\beta J S q a} - 1}$$

↳ dominated by small $|\vec{q}|$, so $\sqrt{1-\gamma_{\vec{q}}^2} \approx |\vec{q}|\, a/\sqrt{3}$

$$= M_{GS}^{NIM} - \frac{\sqrt{3}}{2\pi^2} a^2 \underbrace{\int_0^\infty \frac{q\, dq}{e^{-2\sqrt{3}\beta J S q a} - 1}}_{\frac{\pi^2}{72 a^2 \beta^2 J^2 S^2} \text{ if } \mathrm{Re}(a\beta J S) < 0}$$

$$= M_{GS}^{NIM} - \frac{\sqrt{3}}{144} \frac{1}{(\beta J S)^2}$$

⇒ so, it decays $\propto T^2$ instead of $T^{3/2}$ for ferromagnets

4° Dangers of truncated Holstein-Primakoff transformation