Problems in Physics with Many Scales of Length

Physical systems as varied as magnets and fluids are alike in having fluctuations in structure over a vast range of sizes. A novel method called the renormalization group has been invented to explain them

by Kenneth G. Wilson

One of the more conspicuous properties of nature is the great diversity of size or length scales in the structure of the world. An ocean, for example, has currents that persist for thousands of kilometers and has tides of global extent; it also has waves that range in size from less than a centimeter to several meters; at much finer resolution seawater must be regarded as an aggregate of molecules whose characteristic scale of length is roughly 10^{-8} centimeter. From the smallest structure to the largest is a span of some 17 orders of magnitude.

In general, events distinguished by a great disparity in size have little influence on one another; they do not communicate, and so the phenomena associated with each scale can be treated independently. The interaction of two adjacent water molecules is much the same whether the molecules are in the Pacific Ocean or in a teapot. What is equally important, an ocean wave can be described quite accurately as a disturbance of a continuous fluid, ignoring entirely the molecular structure of the liquid. The success of almost all practical theories in physics depends on isolating some limited range of length scales. If it were necessary in the equations of hydrodynamics to specify the motion of every water molecule, a theory of ocean waves would be far beyond the means of 20th-century science.

A class of phenomena does exist, however, where events at many scales of length make contributions of equal importance. An example is the behavior of water when it is heated to boiling under a pressure of 217 atmospheres. At that pressure water does not boil until the temperature reaches 647 degrees Kelvin. This combination of pressure and temperature defines the critical point of water, where the distinction between liquid and gas disappears; at higher pressures there is only a single, undifferentiated fluid phase, and water cannot be made to boil no matter how much the temperature is raised. Near the critical point water develops fluctuations in density at all possible scales. The fluctuations take the form of drops of liquid thoroughly interspersed with bubbles of gas, and there are both drops and bubbles of all sizes from single molecules up to the volume of the specimen. Precisely at the critical point the scale of the largest fluctuations becomes infinite, but the smaller fluctuations are in no way diminished. Any theory that describes water near its critical point must take into account the entire spectrum of length scales.

Multiple scales of length complicate many of the outstanding problems in theoretical physics and in certain other fields of study. Exact solutions have been found for only a few of these problems, and for some others even the best-known approximations are unsatisfactory. In the past decade a new method called the renormalization group has been introduced for dealing with problems that have multiple scales of length. It has by no means made the problems easy, but some that have resisted all other approaches may yield to this one.

The renormalization group is not a descriptive theory of nature but a general method for constructing theories. It can be applied not only to a fluid at the critical point but also to a ferromagnetic material at the temperature where spontaneous magnetization first sets in, or to a mixture of liquids at the temperature where they become fully miscible, or to an alloy at the temperature where two kinds of metal atoms take on an orderly distribution. Other problems that have a suitable form include turbulent flow, the onset of superconductivity and of superfluidity, the conformation of polymers and the binding together of the elementary particles called quarks. A remarkable hypothesis that seems to be confirmed by work with the renormalization group is that some of these phenomena, which superficially seem quite distinct, are identical at a deeper level. For example, the critical behavior of fluids, ferromagnets, liquid mixtures and alloys can all be described by a single theory.

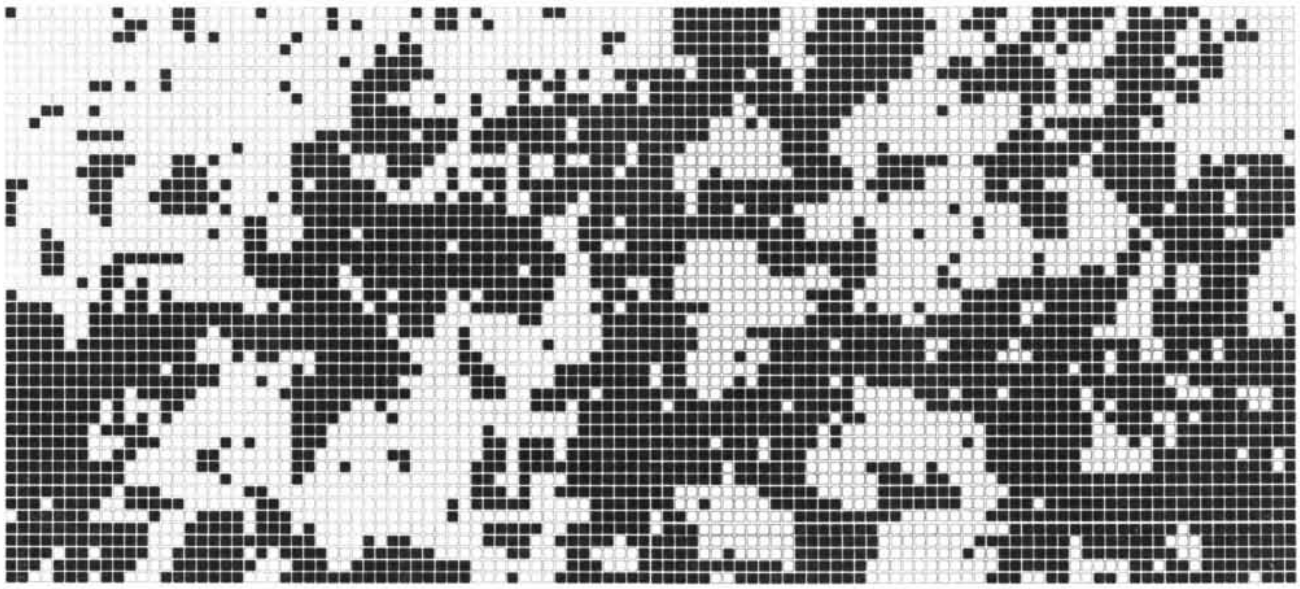
The most convenient context in which to discuss the operation of the renormalization group is a ferromagnet, or permanent magnet. Ferromagnetic materials have a critical point called the Curie point or the Curie temperature, after Pierre Curie, who studied the thermodynamics of ferromagnets at about the turn of the century. For iron the Curie temperature is 1,044 degrees K. At higher temperatures iron has no spontaneous magnetization. As the iron is cooled the magnetization remains zero until the Curie temperature is reached, and then the material abruptly becomes magnetized. If the temperature is reduced further, the strength of the magnetization increases smoothly.

Several properties of ferromagnets besides the magnetization behave oddly

MULTIPLE SCALES OF LENGTH characterize the patterns that emerge when a ferromagnetic solid is cooled to the temperature at which it becomes spontaneously magnetized. Each square represents the magnetic moment associated with a single atom in the solid, and each moment is assumed to have only two possible orientations, labeled "up" (*black squares*) and "down" (*open squares*). At high temperature (*top*) the orientation of the magnetic moments is essentially random, and so there is only short-range order in the pattern. As the temperature is reduced (*middle*) somewhat larger patches in which most of the magnetic moments are lined up in the same direction begin to develop. When the temperature reaches a critical value called the Curie temperature, or T_c (*bottom*), these patches expand to infinite size; significantly, however, fluctuations at smaller scales persist. As a result all scales of length must be included in a theoretical description of the ferromagnet. This simulation of a ferromagnet was carried out with the aid of a computer by Stephen Shenker and Jan Tobochnik of Cornell University.

$T = 2T_c$

$T = 1.05T_c$



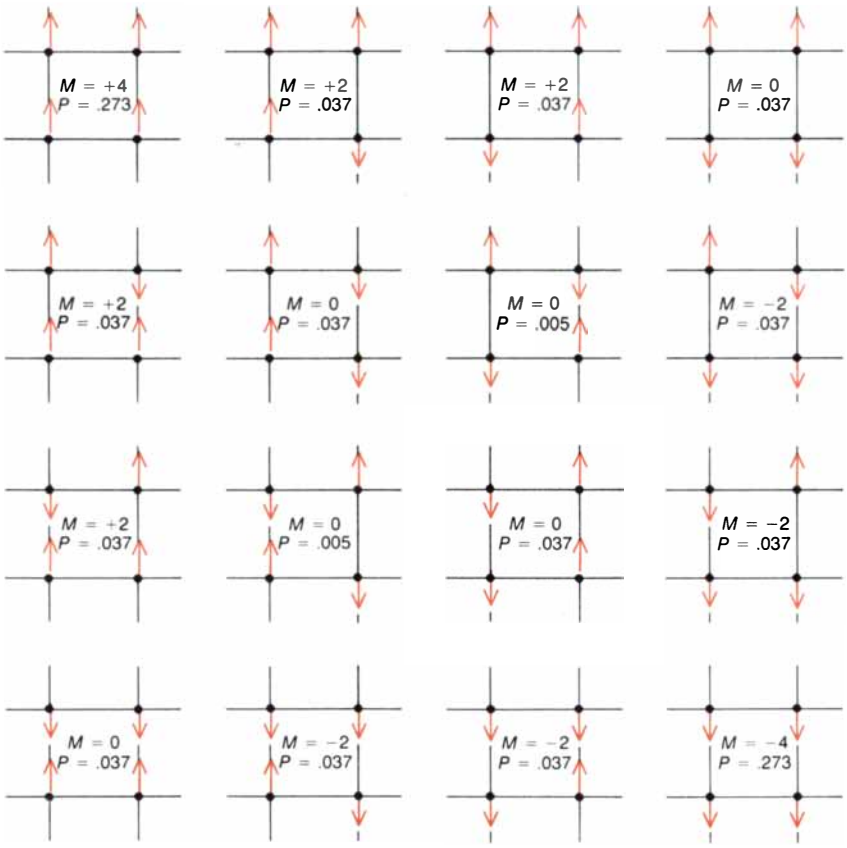
$T = T_c$

near the Curie point. Another property of interest is the magnetic susceptibility, or the change in magnetization induced by a small applied field. Well above the Curie point the susceptibility is small because the iron cannot retain any magnetization; well below the Curie temperature the susceptibility is small again because the material is already magnetized and a weak applied field cannot change the state of the system very much. At temperatures close to 1,044 degrees, however, the susceptibility rises to a sharp peak, and at the Curie point itself the susceptibility becomes infinite.

The ultimate source of ferromagnetism is the quantum-mechanical spinning of electrons. Because each electron rotates it has a small magnetic dipole moment; in other words, it acts as a magnet with one north pole and one south pole. How the spin of the electron gives rise to the magnetic moment will not concern me here. It is sufficient to note that both the spin and the magnetic moment can be represented by a vector, or arrow, which defines the direction of the electron's magnetic field.

A real ferromagnet has a complex atomic structure, but all the essential properties of the system of spins can be illustrated by a quite simple model. Indeed, I shall describe a model that includes no atoms or other material particles but consists only of spin vectors arranged in a lattice. For the sake of simplicity I shall deal with a two-dimensional lattice: a rectilinear grid of uniformly spaced lines in a plane, with a spin vector at each intersection of the grid lines. Furthermore, I shall assume that each spin can point in only two possible directions, designated up and down. The model lattice is said to be magnetized whenever more than half of the spins point in the same direction. The magnetization can be defined as the number of up spins minus the number of down spins.

Every electron has the same spin and the same magnetic dipole moment. What distinguishes a ferromagnet from other materials is a coupling between nearby spins that makes them tend to line up in the same direction. This tendency can be stated more precisely by pointing out that the total energy of any two adjacent spins is smaller when the spins are parallel than it is when they are antiparallel. The interaction responsible for the coupling of the spins has a short range, which is reflected in the model by specifying that only nearest-neighbor spins are coupled to each other. In the two-dimensional rectilinear lattice each spin is influenced by four nearest neighbors; no other spins have any direct effect on it.



MODEL OF A FERROMAGNET consists of vectors, or arrows of fixed length, arranged at the sites of a lattice. Each vector represents the spin angular momentum and the magnetic moment of a single electron, and it can be oriented either up or down. Nearest-neighbor lattice sites are coupled in such a way that adjacent spin vectors are likelier to be parallel than antiparallel. From the strength of the coupling, which declines as the temperature increases, a probability, *P*, can be assigned to every possible configuration of the spin vectors. All the configurations of a lattice made up of just four sites are shown here. The net magnetization, *M*, of each configuration is easily calculated: it is the number of up spins minus the number of down spins. The magnetization of the model at any given temperature is found by multiplying the magnetization of each configuration by the probability of that configuration, then adding up all the results. The probabilities shown were calculated for a coupling strength of .5, which corresponds to a temperature (in arbitrary units) of 2. The model is called the two-dimensional Ising model.

From the nature of the interaction between spins in a ferromagnet one might well predict that all the spins would always be parallel and the material would always have its maximum magnetization. That is the state of lowest energy, and in the absence of any perturbing effects it would be the favored state. In a real ferromagnet, however, there is one perturbation that cannot be neglected: the thermal motion of the atoms and the electrons. At any temperature above absolute zero thermal excitations of the solid randomly flip some of the spins so that the direction of the spin vector is reversed, even when reversing the spin puts the magnet in a state of higher energy. Hence it is no surprise that the magnetization decreases as the temperature increases: that relation simply reflects increasing thermal disruption. What remains curious is that the magnetization is not a smooth function of the temperature but instead disappears abruptly at a certain finite temperature, the Curie point.

Competition between the tendency toward a uniform spin orientation and the thermal introduction of disorder can readily be incorporated into a model of a ferromagnet. The strength of the coupling between adjacent spins is given by a number, K, that must be specified in the design of the model. Thermal effects are included simply by making K inversely proportional to the temperature. With the appropriate units of measurement the coupling strength can be set equal to the reciprocal of the temperature, a relation expressed by the equation $K = 1/T$.

What the coupling strength determines is the probability that two adjacent spins are parallel. When the temperature is zero, there are no thermal effects and adjacent spins are certain to be parallel; the probability is equal to 1 and the coupling strength is infinite. At an infinite temperature the coupling strength falls to zero, so that the spins do not interact at all. Hence each spin is free to choose its direction randomly and is independent of its neighbors. The probability that two spins are parallel is 1/2, and so is the probability that they are antiparallel. The region of interest,

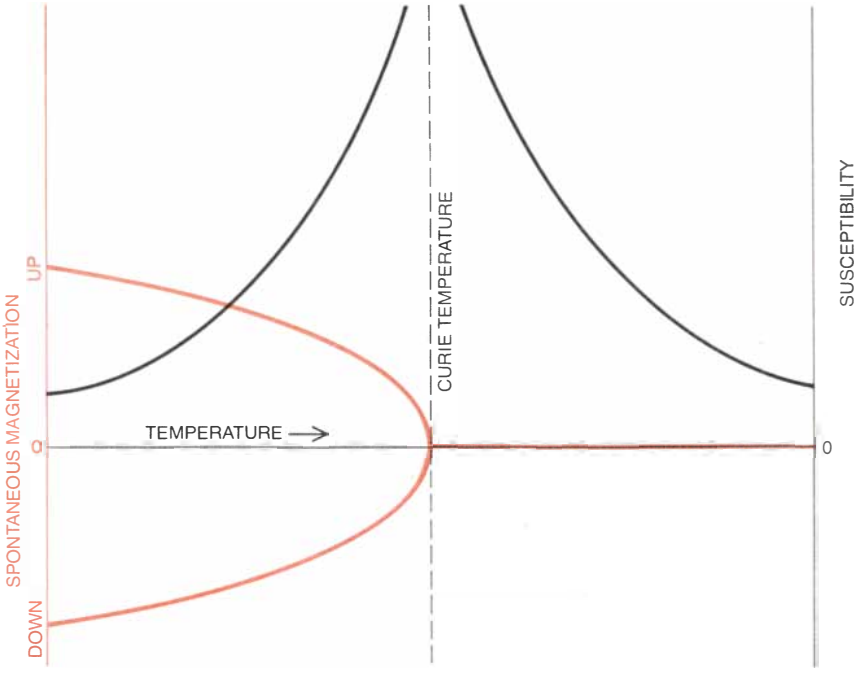
of course, lies between these extremes of temperature, where the probability of the adjacent spins' lining up must always have a value between 1/2 and 1.

Suppose there is a large two-dimensional lattice of spins and that some one spin in it is artificially held fixed in the up orientation. What is the effect on the other spins? The effect on the spins at the four adjacent lattice sites is easy to imagine: since they are directly coupled to the fixed spin, they will have a greater-than-even probability of pointing up. The extent to which the probability is biased depends on the value of K, which is determined in turn by the temperature.

More distant spins have no direct interaction with the fixed spin, but nonetheless the influence of the fixed spin does not end with the immediate neighbors. Because the nearest-neighbor spins tend to point up more often than down they create a similar bias in their own nearest neighbors. In this way the disturbance can propagate over a large area of the lattice. The range of influence of a single fixed spin can be measured by observing the orientation of many spins that are all at the same large distance from the fixed one. If reversing the orientation of the fixed spin from up to down increases the number of down spins in the distant population, then the spins are said to be correlated. The maximum distance over which such a correlation can be detected is called the correlation length. Regions separated by a distance greater than the correlation length are essentially independent.

In a lattice at very high temperature the correlation length is close to zero. The distribution of spins is nearly random, and so the average number of up and down spins must be equal; in other words, the magnetization is zero. As the temperature falls (and the coupling strength increases) correlations over larger distances begin to appear. They take the form of spin fluctuations, or patches of a few spins each that mostly point in the same direction. Over any large area the magnetization is still zero, but the structure of the lattice is much different from what it was near infinite temperature.

As the temperature approaches the Curie point the correlation length grows rapidly. The basic interactions of the model have not changed; they still connect only adjacent lattice sites, but long-range order has emerged from the short-range forces. What is most significant in the growth of the correlation length is that as the maximum size of the spin fluctuations increases, the smaller fluctuations are not suppressed; they merely become a finer structure superimposed on the larger one. The largest fluctuations are not areas of uniform spin alignment; they include many smaller fluctuations and can be distinguished only because they have an overall excess of one spin direction. Thus an ocean of spins that are mostly up may have within it an island of spins that are mostly down, which in turn surrounds a lake of up spins with an islet of down spins. The progression continues to the smallest possible scale: a single spin.



MAGNETIZATION of a ferromagnet has a sudden onset at the Curie temperature. Above this temperature the average numbers of up spins and down spins are equal and so the magnetization is zero. At any temperature below the Curie point two states of magnetization are possible, depending on whether the up spins or the down spins are in the majority; in the absence of an external magnetic field the two states are equally likely. The susceptibility of a ferromagnet measures the change in magnetization induced by an arbitrarily small applied magnetic field. At the Curie point the susceptibility becomes infinite. Near the Curie point a small change in either the temperature or the external field gives rise to a large change in the magnetization.

When the temperature is precisely equal to the Curie temperature, the correlation length becomes infinite. Any two spins are correlated, no matter what the distance between them is. Nevertheless, fluctuations persist at all smaller scales of length. The system remains unmagnetized, but it is exquisitely sensitive to small perturbations. For example, holding a single spin fixed in the up orientation creates a disturbance that spreads throughout the lattice and gives the entire system a net magnetization.

Below the Curie temperature the system becomes magnetized even in the absence of an outside perturbation, but there is no immediate change in the appearance of the lattice. Smaller-scale fluctuations persist; they are remnants of the lakes and islets of opposite spin direction. Merely by looking at the lattice one cannot detect the magnetization. Only when the system is cooled further does the bias become obvious, as the increasing coupling strength coerces more of the spins into conformity with the majority. At zero temperature complete uniformity is attained.

In fluids the fluctuations in density near the critical point are closely analogous to the fluctuations in spin direction observed in ferromagnets. In fluids, however, the presence of fluctuations at all possible scales of length can be observed directly. When the correlation length first reaches a few thousand angstrom units, which is comparable to the wavelength of light, the fluctuations begin to scatter light strongly and the fluid turns milky, a phenomenon called critical opalescence. Significantly, when the temperature comes still closer to the critical point and the maximum scale of the fluctuations becomes much larger (millimeters or centimeters), the critical opalescence is not reduced, indicating that the smaller fluctuations persist. The same phenomenon takes place in spin systems, but because ferromagnetic materials are not transparent to light it cannot be demonstrated as readily. The critical opalescence of ferromagnets has been detected, however, in the scattering of neutrons from a magnetic material near the Curie temperature.

The model I have been describing is not my own invention. It is a version of one introduced in the 1920's by the German physicists Wilhelm Lenz and Ernest Ising, and it is now called the Ising model. The properties of a system of Ising spins on a two-dimensional lattice are known in complete detail because the model was solved exactly in 1944 by Lars Onsager of Yale Universi-

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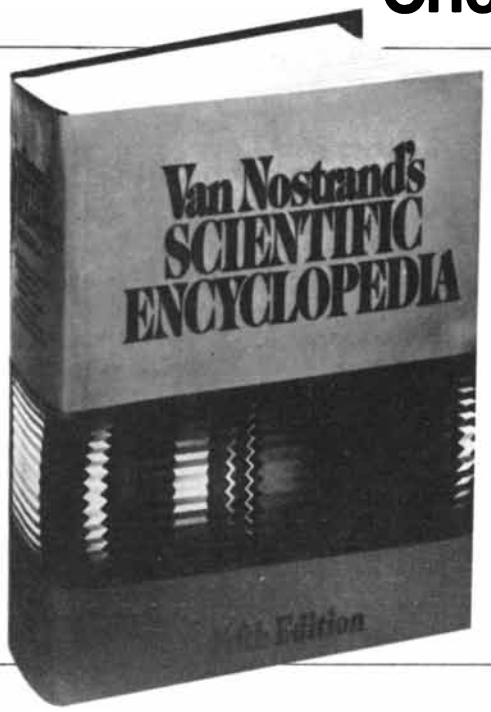

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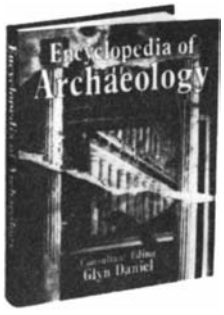

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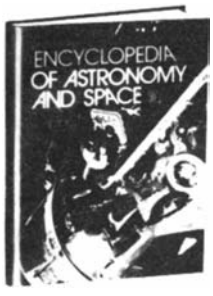

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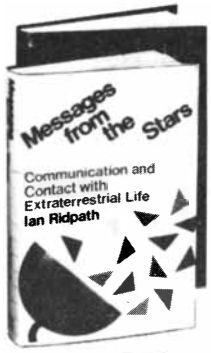

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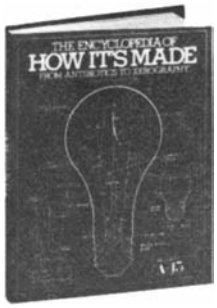

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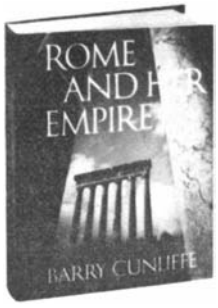

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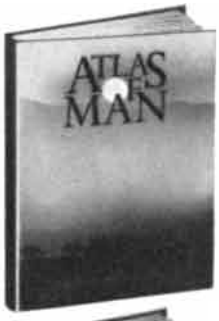

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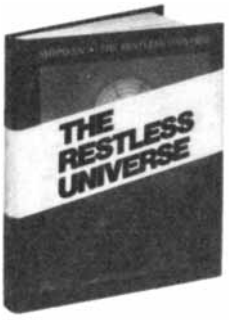

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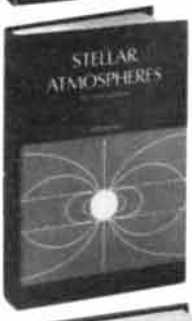

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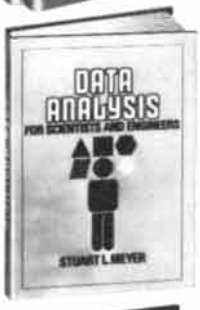

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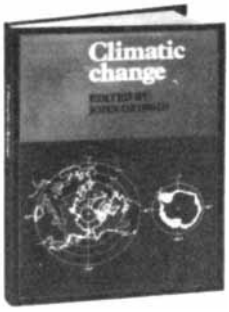

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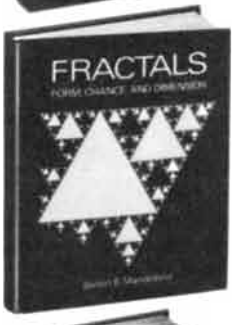

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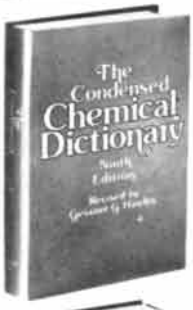

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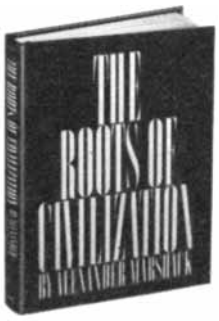

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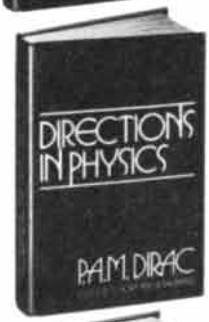

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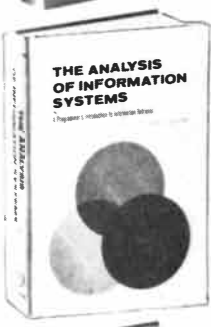

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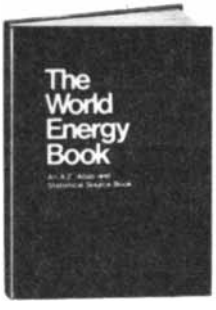

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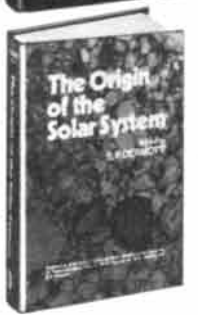

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ty. Since then solutions have also been found for several other two-dimensional models (whereas no three-dimensional model has yet been solved exactly). Nevertheless, the problems of describing two-dimensional systems are far from trivial. In what follows I shall apply the methods of the renormalization group to the two-dimensional Ising model as if it were a problem still outstanding, and Onsager's solution will serve as a check on the results.

What does it mean to solve or to understand a model of a physical system? In the case of the Ising system the microscopic properties are known completely from the outset, since they were specified in building the model. What is needed is a means of predicting the macroscopic properties of the system from the known microscopic ones. For example, a formula giving the spontaneous magnetization, the susceptibility and the correlation length of the model as a function of temperature would contribute greatly to understanding.

It is not notably difficult to calculate the macroscopic properties of any given configuration of the spins in an Ising model. The magnetization, for example, can be determined simply by counting the number of up spins and the number of down spins and then subtracting. No one configuration of the spins, however, determines the macroscopic properties of the system. Instead all possible configurations contribute to the observed properties, each in proportion to its probability at a given temperature.

In principle the macroscopic properties could be calculated directly as the sum of all the separate contributions. First the magnetization would be found for each configuration and then the corresponding probability. The actual magnetization would be obtained by multiplying each of these pairs of numbers and adding up all the results. The susceptibility and the correlation length could be found by procedures that are not much more elaborate. The common element in all these calculations is the need to determine the probabilities of all possible configurations of the spins. Once the distribution of probabilities is known the macroscopic properties follow directly.

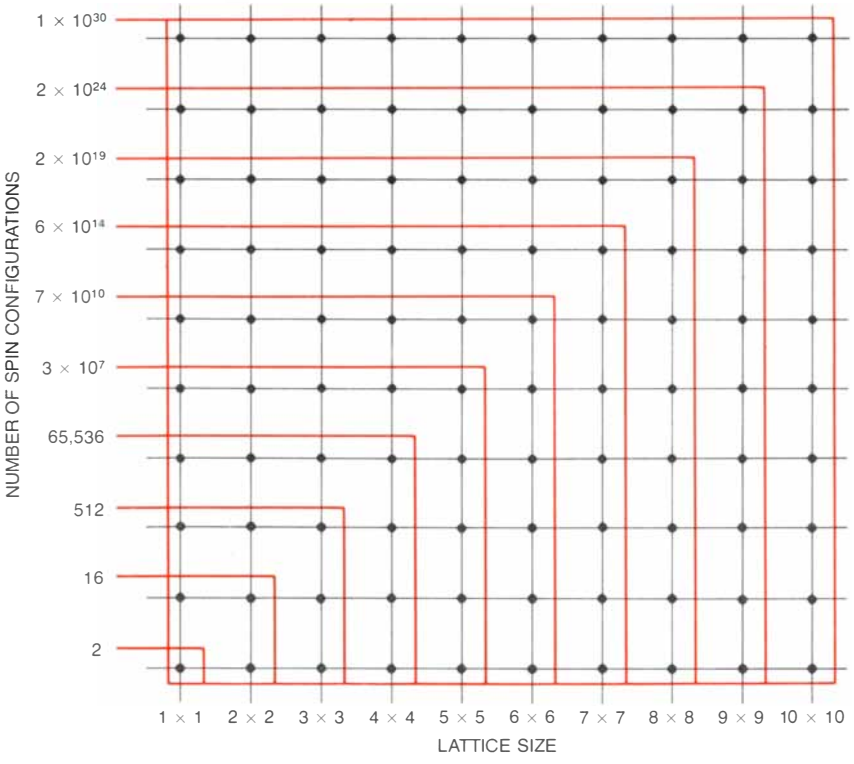
As I pointed out above, the probability of any two adjacent spins' being parallel is determined solely by the coupling strength K, which I have defined as the reciprocal of the temperature. If the probability of two neighboring spins in isolation being parallel is denoted p, then the probability of their being antiparallel must be $1 - p$. From these two values alone the relative probability of any specified configuration of a lattice can be evaluated. All that is required is to multiply together the separate probabilities for every nearest-neighbor pair of spins, in each case taking the value as p when the spins are parallel and as $1 - p$ when they are antiparallel.

Consider a spin system that is made up of just four spins arranged at the corners of a square. Such a lattice has four nearest-neighbor couplings, corresponding to the four sides of the square. Each coupling is considered in turn and is assigned a probability of either p or $1 - p$ according to whether the spins are parallel or antiparallel; then the four separate probabilities are multiplied. In the configuration with all four spins oriented up all four pairs are parallel, and so the relative probability is given by the product $p \times p \times p \times p$. If three spins are up and one is down, the relative probability is $p \times p \times (1 - p) \times (1 - p)$.

The calculation must be carried out for every configuration of the spins; for a system of four spins there are 16 configurations. A final step is to convert the relative probabilities into absolute ones by adjusting each value so that the total of all 16 values is equal to exactly 1. Since the temperature determines the coupling strength and the coupling strength in turn determines the values of p and $1 - p$, the entire sequence of 16 calculations would also have to be repeated for every temperature of interest.

This plan of attack on the Ising model is ambitious but impractical. If the probability of every spin configuration could be calculated, the magnetization and the other macroscopic properties could be evaluated for any specified temperature. The problem lies in the number of spin configurations. For a system made up of n spins, each of which can take on two values, there are 2^n possible configurations. This exponential function grows rapidly as n increases. As I have mentioned, four spins have 2^4, or 16, configurations. A three-by-three block of nine spins has 512 configurations and a four-by-four block has 65,536. The practical limit of computation is not much larger than a six-by-six block of 36 spins, for which there are approximately 7×10^{10} configurations.

What size lattice would be needed in order to determine the critical properties of the two-dimensional Ising model? The array must be at least as large as the largest fluctuations observed at the temperature of interest. At a temperature reasonably close to the Curie point the correlation length, in units of the lattice spacing, might be about 100 and the largest fluctuations would cover about 100^2, or 10,000, lattice sites. A block of spins that large has $2^{10,000}$ possible configurations, a number that is somewhat greater than $10^{3,000}$. The fastest computer conceivable could not carry out such a calculation. Even if the computer had been working continuously since the



NUMBER OF SPIN CONFIGURATIONS rises steeply as the size of a lattice grows. For a system of n spins, each of which has two possible values, the number of configurations is equal to 2^n. When the lattice is large, it becomes impractical to calculate the probability of all the configurations. The limit of practical computation is a lattice somewhat larger than the six-by-six array of 36 spins. In order to observe the critical behavior of the system near the Curie temperature an array of about 100-by-100 spins would be needed, which has $2^{10,000}$ configurations.

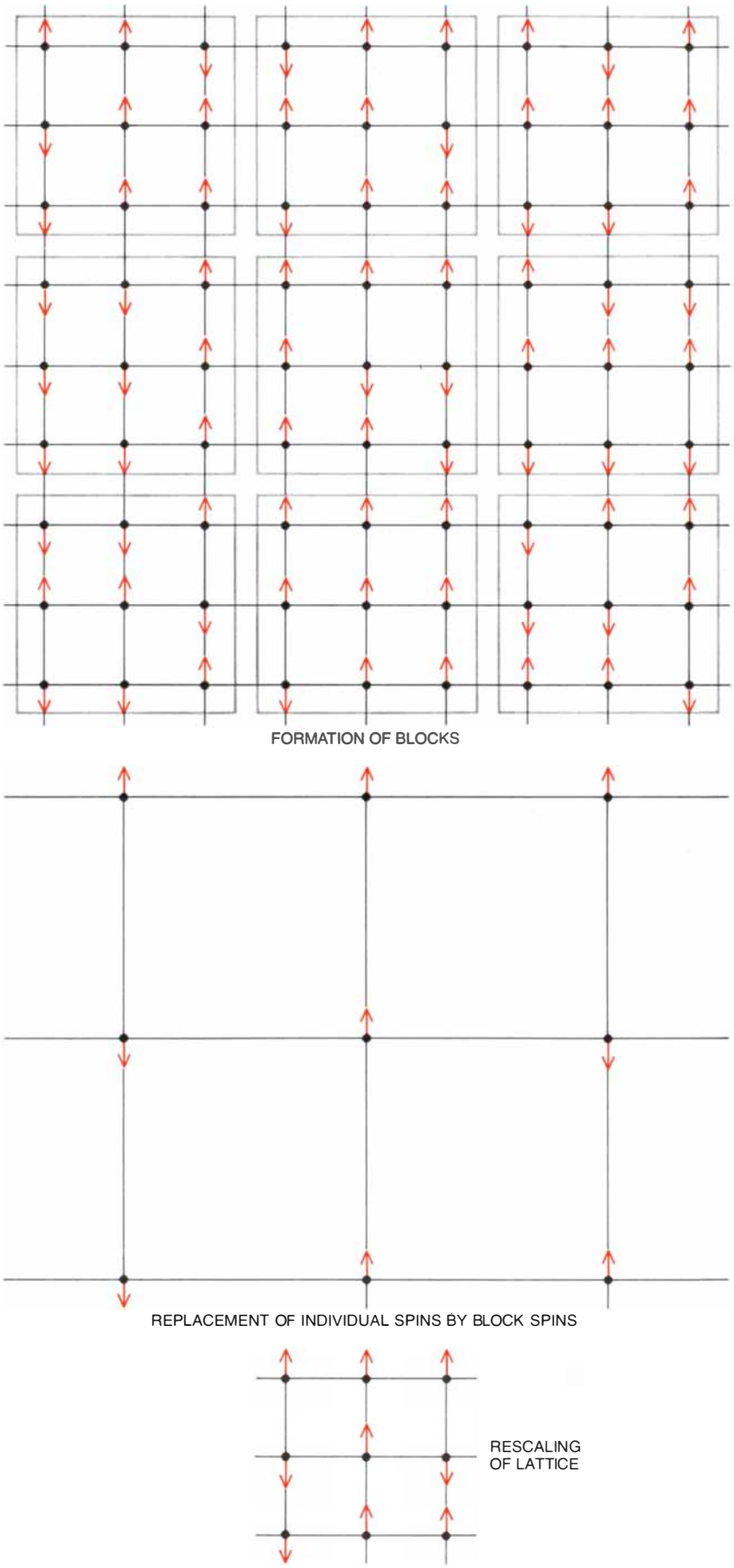
"big bang" with which the universe began, it would not yet have made a significant start on the task.

The need to carry out an almost endless enumeration of spin configurations can be circumvented for two special conditions of the lattice. When the temperature of the system is zero (so that the coupling strength is infinite), all but two of the configurations can be neglected. At zero temperature the probability that a pair of spins will be antiparallel falls to zero, and therefore so does the probability of any configuration that includes even one antiparallel pair. The only configurations that do not have at least one antiparallel pair are those in which all the spins are up or all are down. The lattice is certain to assume one of these configurations, and all other configurations have zero probability.

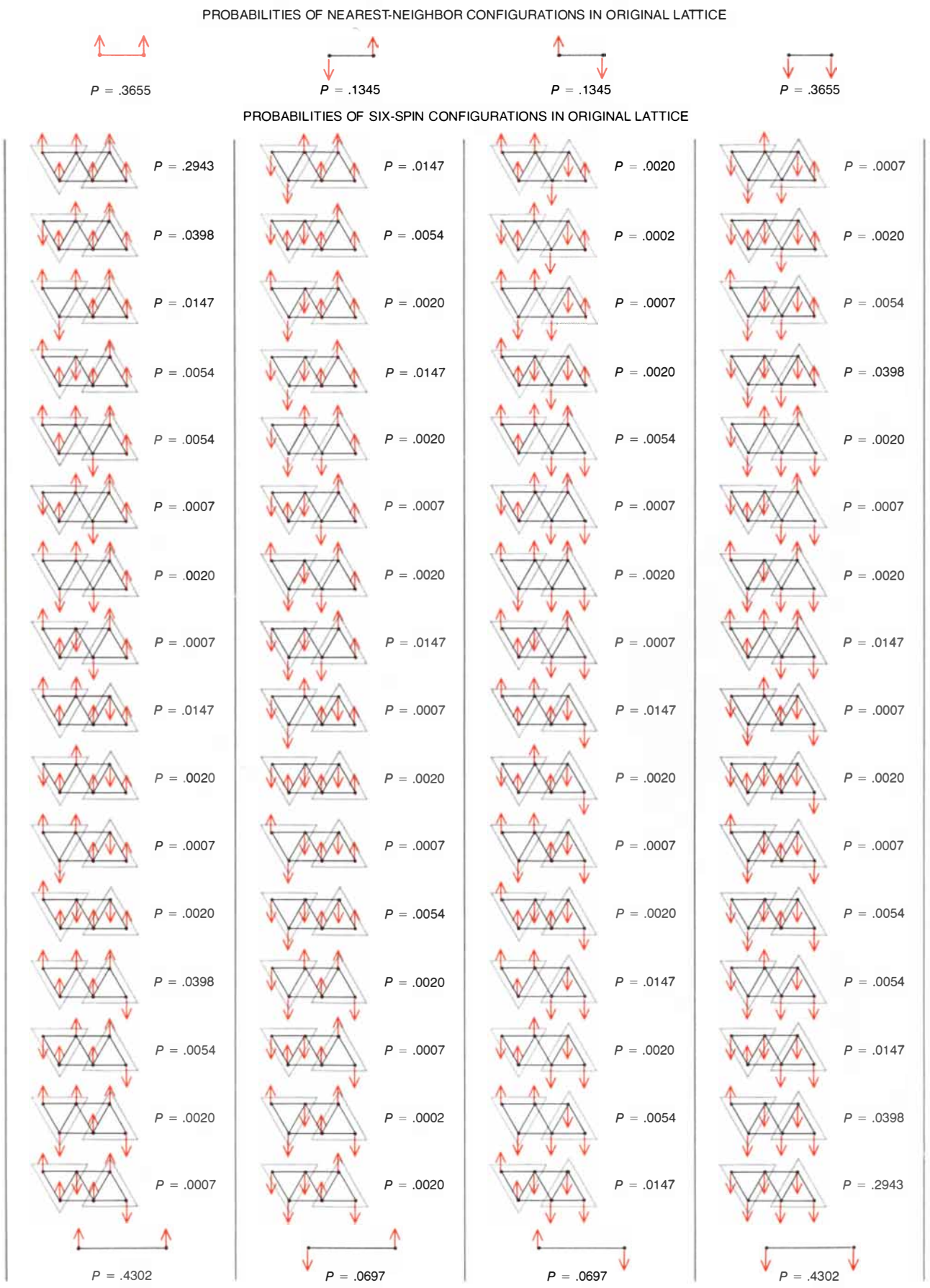
At infinite temperature, where the coupling strength is zero, the probability distribution is also much simplified. Every spin is then independent of its neighbors and its direction at any instant can be chosen at random. The result is that every configuration of the lattice has equal probability.

Through these two shortcuts to the determination of the probability distribution it is a trivial exercise to calculate exactly the properties of the Ising model at absolute zero and at infinite temperature. Acceptable methods of approximation are also available for any temperature low enough to be considered close to zero or high enough to be considered close to infinity. The troublesome region is between these extremes; it corresponds to the region of the critical point. Until recently there was no practical and direct method of calculating the properties of a system arbitrarily close to the critical point. The renormalization group provides such a method.

The essence of the renormalization-group method is to break a large problem down into a sequence of smaller and more manageable stages. Instead



RENORMALIZATION-GROUP approach to a model ferromagnet consists in breaking down an intractable problem with multiple scales of length into a sequence of smaller problems, each of which is confined to a single scale of length. One version of the renormalization-group method, called the block-spin transformation, has three steps. First the lattice is divided into blocks of a few spins each, in this case nine. Then each block is replaced by a single spin whose value is the average of all the spins in the block; here the average is determined by majority rule. In this way a new lattice is created, with three times the original lattice spacing and one-third the density of spins. Finally the original scale is restored by reducing all dimensions by a factor of 3. The procedure must be carried out for all configurations of a few spins in the original lattice, so that a probability can be found for every configuration of the block spins.



of keeping track of all the spins in a region the size of the correlation length, the long-range properties are deduced from the behavior of a few quantities that incorporate the effects of many spins. There are several ways to do this. I shall describe one, the block-spin technique, in which the principles of the method are revealed with particular clarity. It was introduced by Leo P. Kadanoff of the University of Chicago and was made a practical tool for calculations by Th. Niemeijer and J. M. J. van Leeuwen of the Delft University of Technology in the Netherlands.

The method has three basic steps, each of which must be repeated many times. First the lattice is divided into blocks of a few spins each; I shall employ square blocks with three spins on a side, so that each block includes nine spins. Next all the spins in the block are averaged in some way and the entire block is replaced by a single new spin with the value of the average. Here the averaging can be done by a simple procedure: by following the principle of majority rule. If five or more of the original spins are up, the new spin is also up; otherwise it is down.

The result of these two operations is to create a new lattice whose fundamental spacing is three times as large as that of the old lattice. In the third step the original scale is restored by reducing all dimensions by a factor of 3.

These three steps define a renormalization-group transformation. Its effect is to eliminate from the system all fluctuations in spin direction whose scale is smaller than the block size. In the model given here any fluctuation of the spins over a range of fewer than three lattice units will be smeared out by the averaging of the spins in each block. It is as if one looked at the lattice through an out-of-focus lens, so that the smaller features are blurred but the larger ones are unaffected.

It is not enough to carry out this procedure for any one configuration of the original lattice; once again what is sought is a probability distribution. Suppose one considers only a small region of the initial lattice, consisting of 36 spins that can be arranged in four blocks. The spins in this region have 2^{36}, or about 70 billion, possible configurations. After the block-spin transformation has been applied the 36 original spins are replaced by four block spins with a total of 16 configurations. It is just within the limit of practicality to compute the probability of each of the configurations of the original 36 spins. From those numbers the probabilities of the 16 block-spin configurations can readily be determined. The calculation can be done by sorting all the configurations of the original lattice into 16 classes according to which configuration of the block spins results in each case from applying the principle of majority rule. The total probability for any one configuration of the block spins is then found by adding up the probabilities of all the configurations of the original lattice that fall into that class.

It may well seem that nothing is gained by this procedure. If the complete probability distribution can be calculated for a system of 36 spins, nothing new is learned by condensing that system into a smaller lattice of four block spins. Near the critical point it is still necessary to consider a much larger lattice, with perhaps 10,000 spins instead of 36, and the probability distribution for the block spins generated from this lattice cannot be calculated because there are far too many configurations. As it turns out, however, there is a method for extracting useful information from a small set of block spins. It is a method for observing the behavior of the system over a large region without ever dealing explicitly with the configurations of all the spins in that region.

Each block spin represents nine spins in the original lattice. The complete set of block spins, however, can also be regarded as a spin system in its own right, with properties that can be investigated by the same methods that are applied to the original model. It can be assumed that there are couplings between the block spins, which depend on the temperature and which determine in turn the probability of each possible spin configuration. An initial guess might be that the couplings between block spins are the same ones specified in the original lattice of Ising spins, namely a nearest-neighbor interaction with a strength given by the parameter K, the reciprocal of the temperature.

This guess can easily be checked, because the probability distribution for the configurations of at least a small part of the block-spin system is already known; it was computed from the configurations of the original lattice in the course of defining the block spins. Surprisingly, this hypothesis is generally wrong: the block spins do not have the same couplings as the spins in the original model. Assuming that only adjacent sites interact and that they have a coupling strength equal to K gives the wrong set of probabilities for the configurations of the block spins.

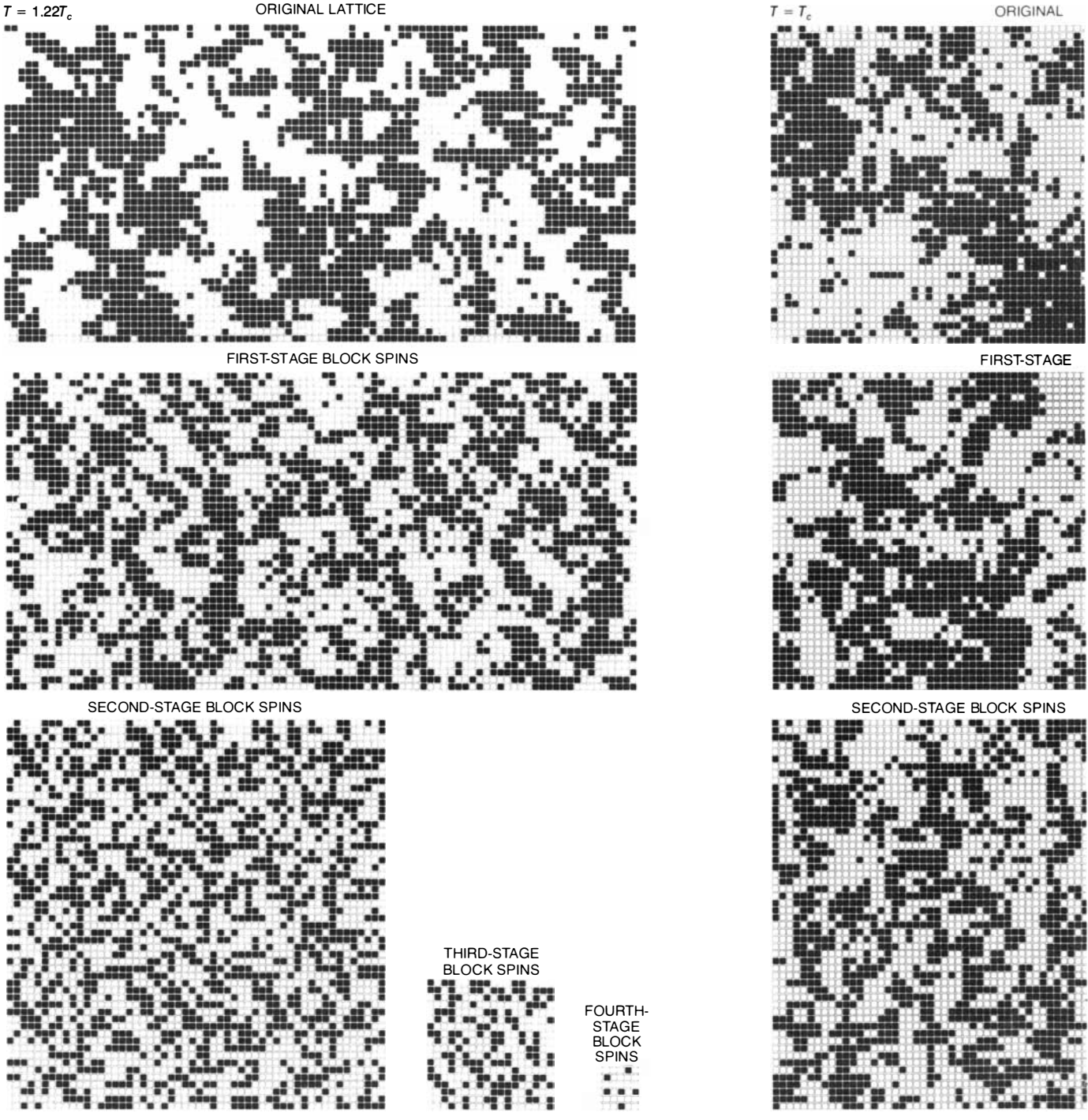
If the specifications of the original model will not describe the system of block spins, then some new set of couplings must be invented. The guiding principle in formulating these new interactions is to reproduce as accurately as possible the observed probability distribution. In general the nearest-neighbor coupling strength must be changed, that is, K must take on a new value. What is more, couplings of longer range, which were excluded by definition from the Ising model, must be introduced. For example, it may be necessary to establish a coupling between spins at the opposite corners of a square. There might also be direct interactions among spins taken three at a time or four at a time. Couplings of still longer range are possible. Hence the block spins can be regarded as a lattice system, but it is a system quite different from the original one. Notably, because the basic couplings have different values, the lattice of block spins is at a temperature different from that of the initial Ising system.

Once a set of couplings has been found that correctly describes the probability distribution for the block spins, a lattice of arbitrary size can be constructed from them. The new lattice is formed the same way the original one was, but now the probability for the spin at each site is determined by the newly derived coupling strengths rather than by the single coupling of the Ising model. The renormalization-group calculation now proceeds by starting all over again, with the new system of block spins as the starting lattice. Once again blocks of nine spins each are formed, and in some small region, such as an array of 36 spins, the probability of every possible configuration is found. This calculation is then employed to define the probability distribution of a second generation of block spins, which are once more formed by majority rule. Examination of the second-generation block spins shows that the couplings have again changed, so that new values must be supplied a second time for each coupling strength. Once the new values have been determined another lattice system (the third generation) can be construct-

PROBABILITY DISTRIBUTION for a system of block spins is found by adding up the probabilities for all the configurations of the original lattice that contribute to each configuration of the block spins. The calculation is shown for a system of six spins on a triangular lattice. Two blocks of three spins each are formed from the lattice, and each block is replaced by a single spin whose orientation is determined by majority rule. The six spins have 64 possible configurations, which are assigned to columns in such a way that all the configurations in each column give rise to the same block-spin configuration. For example, all the configurations in the column at the far left have at least two spins in each block pointing up, so that they are represented by two up block spins. The coupling strength in the original lattice is set equal to .5, which yields the nearest-neighbor probabilities shown at the top of the page. From this set of numbers a probability is calculated for every configuration of the original lattice; then all the probabilities in each column are added up to give the probability of the corresponding block-spin configuration. The block-spin probabilities are not the same as those specified for the original lattice, which implies that the coupling strength is also different, as is the temperature.

ed and the entire procedure can be repeated yet again.

The point of this repetitive operation is that it provides information about the behavior of distinct but related spin systems in which the fundamental scale of length gets larger with each iteration. After the first block-spin transformation the fluctuations at the smallest scale have been eliminated, but those slightly larger, with a scale of roughly three times the original lattice spacing, can be seen more clearly. After the second transformation each block spin represents the 81 spins in a nine-by-nine block of the original lattice, and all fluctuations up to this size range are averaged out, leaving only those larger than nine lattice units. The next iteration removes all fluctuations whose scale is between nine and 27 lattice units, then the following iteration removes those between 27 and 81 units. Eventually fluctuations at all scales up to the correlation length

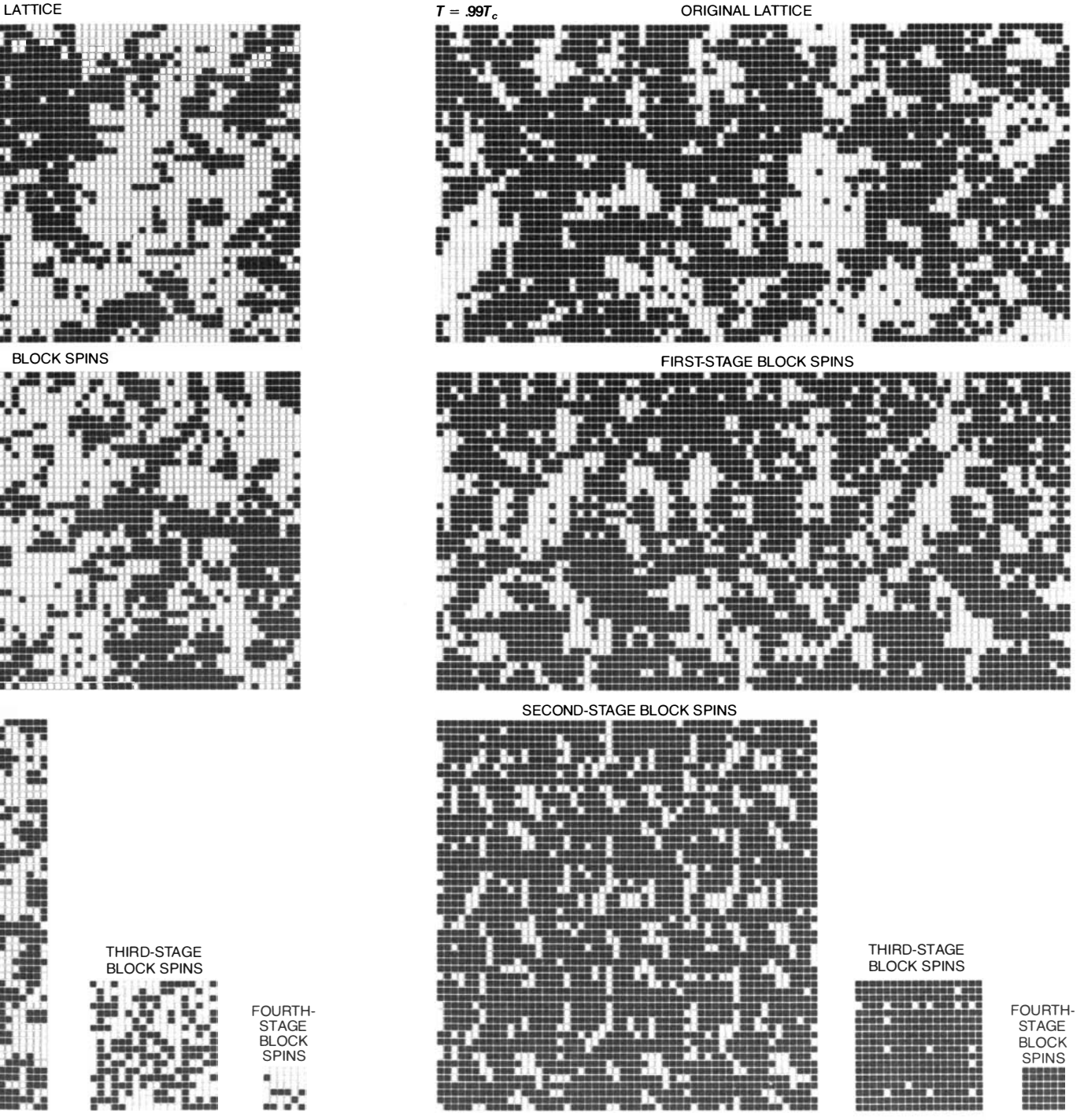


BLOCK-SPIN TRANSFORMATION is applied to a lattice of spins repeatedly, each time elucidating the behavior of the system at a larger scale. The computer simulation, which was carried out by the author, began with an array of some 236,000 spins; a black square represents an up spin and an open square a down spin. The initial temperature was set equal to three values: above the Curie temperature, T_c, at T_c and just below T_c. The transformation begins with the division of the original lattice into three-by-three blocks. Each block is replaced by a single spin whose value is determined by majority rule; these make up the lattice of first-stage block spins. The procedure is then repeated, but with the first-stage block spins serving as the starting lattice. The resulting second-stage spins form the initial configuration for the next transformation, and so on. By the time the third stage is reached the number of spins is small enough for them all

are averaged out. The resulting spin system reflects only the long-range properties of the original Ising system, with all finer-scale fluctuations eliminated.

The value of the block-spin technique can be perceived even through a simple visual inspection of the evolving model. Merely looking at a configuration of Ising spins just below the Curie temperature will seldom reveal that the model is slightly magnetized. At this temperature there is only a small excess of one spin direction over the other, and the many small-scale fluctuations obscure the overall bias. After several applications of the block-spin transformation, however, the smaller fluctuations disappear and the long-range magnetization becomes obvious.

Much of the physical meaning of the block-spin transformation is to be found in the way the couplings between spins change. The rules for deriving the new couplings from the old ones at each



to be shown, and after the fourth stage there are only 36 spins left, each one representing more than 6,000 sites in the original lattice. In the first stage any fluctuations whose scale of length is smaller than three lattice units are eliminated by the averaging procedure. The second stage removes the fluctuations between three and nine lattice units, the third stage those between nine and 27 units, and so on. When the initial temperature is above T_c, the spins become more nearly random in appearance with each iteration and large-scale fluctuations disappear; when the temperature is below T_c, the spins become more nearly uniform and what fluctuations remain are small in scale. When the starting temperature is exactly equal to T_c, large-scale fluctuations remain at all stages. Because the block-spin transformation leaves the large-scale structure of the lattice unchanged at the Curie temperature, a system at that temperature is said to be at a fixed point.

stage are often complicated, but the effect of the change can be illustrated by a quite simple example. Although the assumptions are not realistic, I shall discuss a model in which no couplings with a range longer than the original, nearest-neighbor interaction are introduced. The only change in the coupling is an adjustment in the value of K, which is equivalent to a shift in the temperature. Moreover, this adjustment in K will have a simple form: at each stage in the procedure the coupling strength in the new lattice will be set equal to the square of the coupling in the old lattice. If the new coupling is denoted K', it is given by the equation $K' = K^2$.

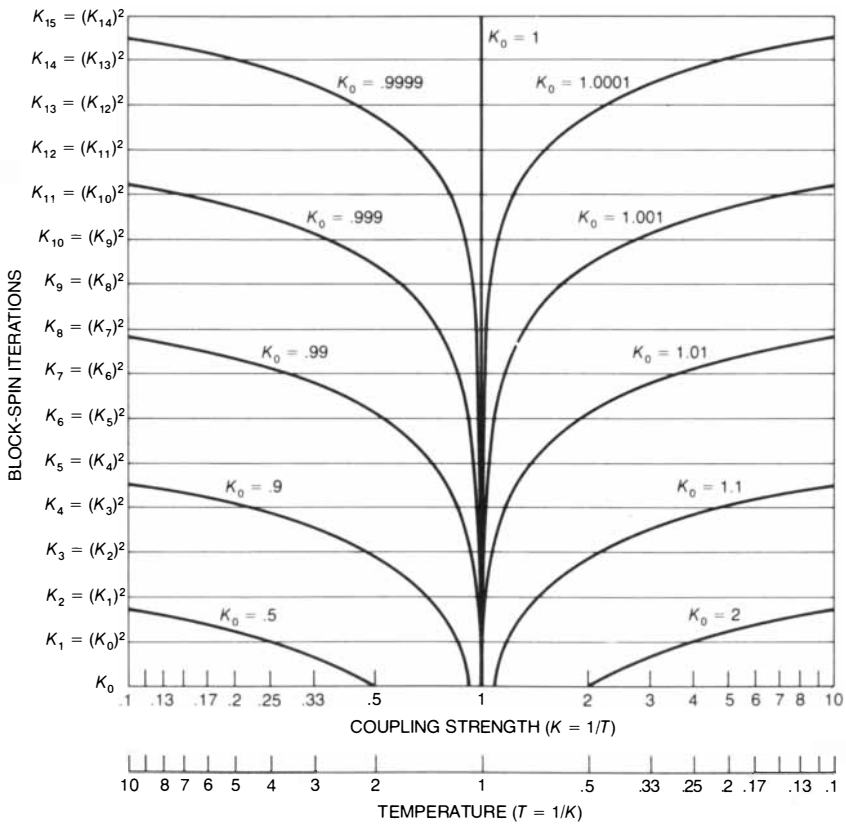
Suppose in some initial state K is equal to 1/2 (which means that the temperature has been given an initial value of 2 in the arbitrary units employed here). In the thinned-out lattice formed as a product of the block-spin transformation K will be replaced by K', with a value of $(1/2)^2$, or 1/4. Repeating the transformation yields successive values of 1/16, 1/256 and so on, in a series that rapidly approaches zero. With each iteration the spin system is converted into a new system that not only has a thinner lattice but also has weaker couplings between the spins. Since K is equal to $1/T$, the temperature increases with each iteration and the lattice approaches the limit of infinite temperature and random spins.

If the initial coupling strength is set equal to 2 (so that the temperature has a value of 1/2), the coupling increases at each stage in the calculation. After the first block-spin transformation the coupling strength is 4, then 16, then 256; ultimately the strength becomes infinite. At the same time, of course, the temperature falls and the system approaches the state of zero temperature, in which all the spins are aligned.

It should be emphasized that what is being observed is not the evolution of any single spin system as the temperature changes. Nothing is being heated or cooled. Instead a new spin system is being created at each stage, a system distinguished by a different set of couplings between spins. The large-scale or long-range behavior of the new lattice is equivalent to the behavior that would be observed in the original lattice at a different temperature.

There is one initial value of K that does not diverge either to infinity or to zero, namely the value $K = 1$. Since 1^2 is equal to 1, K' remains equal to K no matter how many times the transformation is repeated. When K is equal to 1, the system is said to be at a fixed point, where continued application of the renormalization-group transformation leaves all essential properties of the lattice unchanged. Actually the values $K = 0$ and $K =$ infinity also represent fixed points, since zero squared is still zero and infinity squared is still infinity. Zero and infinity, however, are considered trivial fixed points, whereas the value $K = 1$ corresponds to the critical point.

In this discussion of the block-spin technique all the effects of the transformation have been expressed through a single parameter: the nearest-neighbor coupling strength K. Actually many other parameters are introduced by the transformation, each one corresponding to a longer-range coupling. All the possible combinations of these parameters can be represented geometrically by constructing an imaginary multidimensional space in which distance measured along each dimension corresponds to variation in one of the parameters. Every initial state of the spin system and every block-spin transformation of it can be represented by a point on a surface somewhere in this parameter space.



CHANGE IN THE COUPLING BETWEEN SPINS is part of the renormalization-group transformation. The adjustment that must be made to the coupling strength with each iteration can take many forms, but a simple example is presented here: If the strength of the coupling in the original lattice is given by the number K, then in the new lattice the coupling strength is equal to K^2. Any initial value of K greater than 1 must approach infinity when K is squared repeatedly; any value less than 1 must approach zero. The special value $K = 1$ remains unchanged no matter how many times the transformation is repeated. Because the temperature can be defined (in appropriate units) as the reciprocal of the coupling strength, the renormalization-group transformation can be seen as establishing a correspondence between the original lattice and a new, thinned-out lattice that will generally have a different coupling strength and a different temperature. It is only at the fixed point, which corresponds to the Curie temperature, that the coupling and the temperature remain invariant with a value of 1.

In the geometric description of the renormalization-group method the significance of the fixed points becomes apparent. For the two-dimensional Ising system the surface in parameter space has the form of a hilly landscape with two sharp peaks and two deep sinkholes. The ridgeline that connects the peaks and the gully line that connects the sinkholes meet in the center at a saddle point [*see illustration on opposite page*]. One sinkhole is the $K = 0$ fixed point; the other is the $K =$ infinity fixed point. The critical fixed point lies at the point of precarious equilibrium in the saddle.

The transformation of the system from one state to the next can be represented by the motion of a marble rolling on the surface. One can imagine a time-lapse motion picture that would record the marble's position at one-second intervals; then each frame would reveal the effect of one iteration of the block-spin transformation. It is the transformation that allows the marble to move, but the speed and direction of the marble are determined entirely by the slope of the surface at each point it crosses.

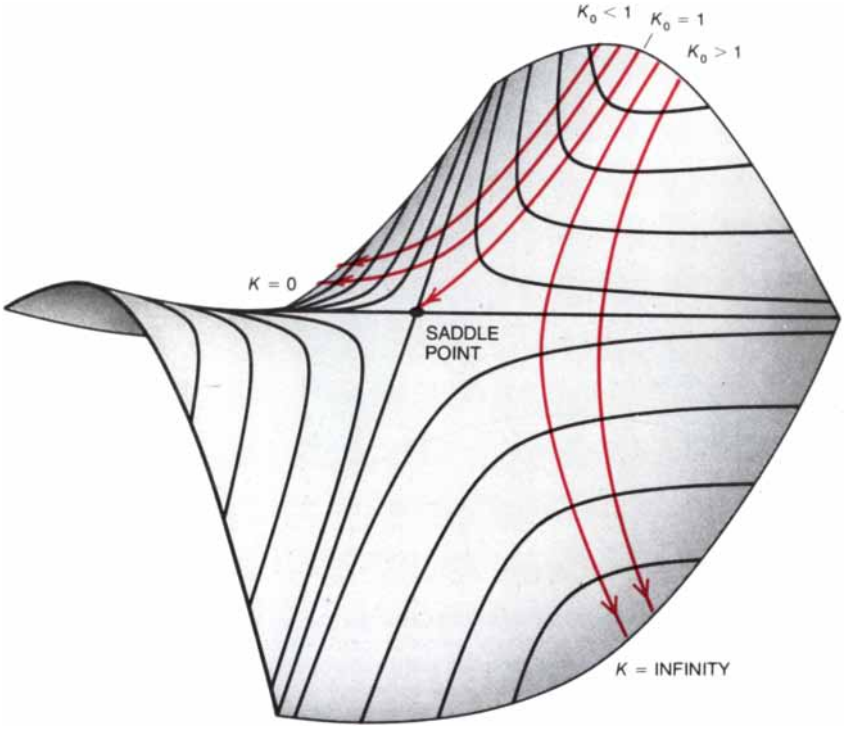
Suppose the marble is initially placed near the top of a hill and just to one side of the ridgeline. At first it moves rapidly,

because the hill is steep near the top, and proceeds in the general direction of the saddle point. As the marble approaches the saddle the slope becomes more gradual and the marble slows down, but it never comes to a complete stop. Moreover, because it started to one side of the ridgeline it does not quite reach the saddle point; instead it is deflected to one side and begins to accelerate again, this time toward a sinkhole.

The trajectory of the marble describes the path followed by the point representing a system of Ising spins as it is transformed repeatedly by the block-spin method. The initial position just off the ridgeline corresponds to an initial value of the various coupling parameters that is equivalent to a temperature just above or just below the critical temperature. In terms of the simplified example described above, with just one parameter, the value of K is either slightly greater than 1 or less than 1. Setting the coupling strength equal to 1 is equivalent to placing the marble exactly on the ridgeline. It then moves directly toward the saddle point, or critical fixed point. Again the motion is rapid at first but becomes slower as the saddle is approached. In this case, however, the marble remains balanced between the two descending slopes. Even after a large number of iterations it remains at the fixed point.

A trajectory on the saddle-shaped surface in parameter space can be made to approach the fixed point as closely as is wished by setting the initial value of K sufficiently close to the critical value. In the example considered here, where the critical value of K is 1, the initial value of K might be .9999, which can be squared several times before it changes appreciably. As a result the trajectory comes quite close to the critical fixed point before it veers off toward the high-temperature sinkhole.

By the examination of many such trajectories the topography of the surface itself can be mapped in the small region surrounding the saddle point. The slope of the surface is what determines how the system approaches the fixed point and how it departs from it. Knowing the slope, then, one can calculate how the properties of the system vary as the initial coupling and the initial temperature are changed. That is precisely the information sought for an understanding of critical phenomena.



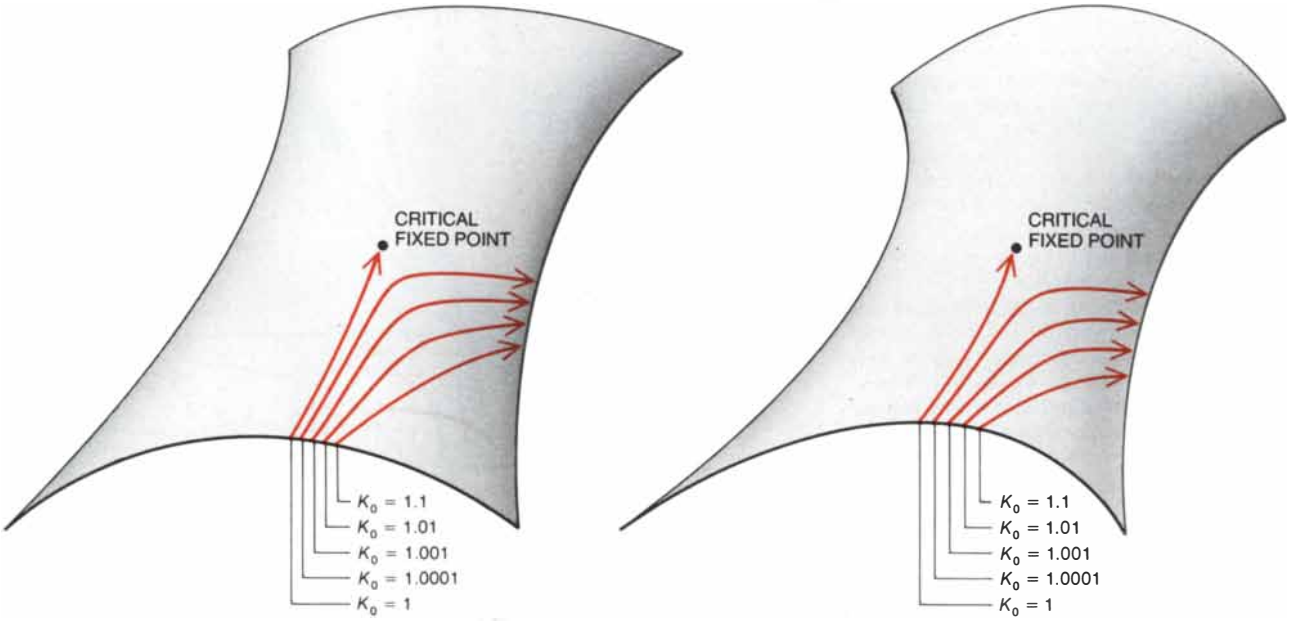
EVOLUTION OF A SPIN SYSTEM in response to repeated renormalization-group transformations can be described as the motion of a point on a surface constructed in an imaginary, multidimensional space: the parameter space. The form of the surface is defined by all the couplings between block spins, but only the nearest-neighbor coupling, K, is considered here. The surface has two peaks and two sinkholes, which are connected to a saddle point. The trajectory followed by the point that represents the state of the system is determined entirely by the slope of the surface. An initial value of K slightly greater than 1 corresponds to an initial position slightly to one side of the ridgeline that connects the peaks. After several block-spin transformations the point rolls down the hill, passes near the saddle point and veers off into one of the sinkholes, where K tends toward infinity. An initial value of K slightly less than 1 leads to a similar trajectory on the other side of the ridgeline and terminates in the other sinkhole, where K approaches zero. When K is equal to exactly 1, the point remains permanently on the ridgeline, approaching equilibrium at the saddle point. Both of the sinkholes are fixed points (since the values of $K = 0$ and $K =$ infinity do not change with further renormalization-group transformations), but they are considered trivial fixed points. The saddle defines the critical fixed point.

The macroscopic properties of a thermodynamic system near the critical point are determined by the temperature. To be more precise, properties such as the spontaneous magnetization, the susceptibility and the correlation length are functions of the amount by which the temperature of the system departs from the critical temperature, T_c. For this reason it is convenient to define the temperature in such a way that all critical points are equivalent. A suitable quantity is the reduced temperature, t, defined as the difference between the actual temperature and the critical temperature, divided by the critical temperature; thus t is equal to $T - T_c/T_c$. On an ordinary temperature scale such as the Kelvin scale the critical temperatures of different systems fall at different values, but all critical points have the same reduced temperature, namely zero.

All critical properties are proportional to the absolute value of the reduced temperature raised to some power. The problem of describing critical phenomena is to determine what that power is, or in other words to determine the values of the critical exponents. For example, the magnetization, M, of a spin system is given by the proportionality $M \quad |t|^{\beta}$, where β (the Greek letter beta) is a critical exponent and where the vertical lines designate the absolute value of t. The magnetic susceptibility is proportional to $1/|t|^{\gamma}$, where γ (the Greek letter gamma) is another exponent. The correlation length is associated with a third exponent, ν (the Greek letter nu), in a relation of the same form: the length is proportional to $1/|t|^{\nu}$.

The earliest attempts to formulate a mathematical description of critical phenomena were theories of a kind that are now called mean-field theories. The first of these was introduced in 1873 by J. D. van der Waals as an explanation of phase changes in fluids. A theory of magnetic phase transitions was proposed in 1907 by Pierre Weiss. In 1937 L. D. Landau of the Academy of Sciences of the U.S.S.R. proposed a more general formulation of mean-field theory, thereby providing a framework in which many physical systems could be discussed. In all these theories the state of any selected particle is determined by the average properties of the material as a whole, properties such as the net magnetization. In effect all particles in the system contribute equally to the force at every site, which is equivalent to assuming that the forces have infinite range.

Mean-field theories are qualitatively successful. They account for important



SLOPE OF THE PARAMETER SURFACE in the vicinity of the critical fixed point determines the macroscopic properties of the Ising model. If trajectories are plotted for many initial values of K near the critical value (which in this case is $K = 1$), it is the slope at the saddle point that determines how quickly the trajectories veer off toward the trivial fixed points at $K = 0$ and $K =$ infinity. If the surface is comparatively flat (*left*), a trajectory with an initial value of K such as $K = 1.01$ passes close to the saddle point. When the surface is more steeply curved (*right*), the corresponding trajectory bends more abruptly toward the sinkhole. Because the temperature is the reciprocal of K the slope near the fixed point reveals how the properties of the system change as the temperature departs from the critical temperature.

features of the phase diagrams of fluids and ferromagnets, the most notable of these features being the existence of a critical point. The quantitative predictions, however, are less satisfactory: the theories give the wrong values for the critical exponents. For β, the exponent that governs the spontaneous magnetization, mean-field theory implies a value of 1/2; in other words, the magnetization varies as the square root of the reduced temperature. The exponent associated with the susceptibility, γ, is assigned a value of 1, so that the susceptibility is proportional to $1/|t|$. The exponent for the correlation length, ν, is 1/2, so that this quantity is proportional to $1/\sqrt{|t|}$.

The exponents calculated from mean-field theory suggest a plausible form for each of these functions. The magnetization has two possible values ($+\sqrt{t}$ and $-\sqrt{t}$) at all temperatures below the critical point, and then it vanishes above the critical temperature. Both the susceptibility and the correlation length approach infinity as t nears zero from either above or below. The actual values of the mean-field exponents, however, are known to be wrong.

For the two-dimensional Ising model the critical exponents are known exactly from Onsager's solution. The correct values are $\beta = 1/8$, $\gamma = 7/4$ and $\nu = 1$, which differ significantly from the predictions of mean-field theory and imply that the system has rather different behavior. For example, the magnetization is proportional not to the square root of the reduced temperature t but to the eighth root of t. Similarly, the susceptibility is given by the reciprocal not of t but of t raised to the 1.75th power, which makes the divergence near the critical point steeper and more abrupt.

The reason for the quantitative failure of mean-field theories is not hard to identify. The infinite range assigned to the forces is not even a good approximation to the truth. Not all spins make equal contributions; the nearest neighbors are more important by far than any other spins. The same objection can be expressed another way: the theories fail to take any notice of fluctuations in spin orientation or in fluid density.

In a renormalization-group calculation the critical exponents are determined from the slope of the parameter surface in the vicinity of the fixed point. A slope is simply a graphic representation of a rate of change; the slope near the fixed point determines the rate at which the properties of the system change as the temperature (or the coupling strength) is varied over some narrow range near the critical temperature. Describing the change in the system as a function of temperature is also the role of the critical exponents, and so it is reasonable that there is a connection between the exponents and the slope.

Renormalization-group calculations for the two-dimensional Ising system have been carried out by several workers. In 1973 Niemeijer and van Leeuwen employed a block-spin method to study the properties of a system of Ising spins constructed on a triangular lattice. I have applied a somewhat different renormalization-group technique, called spin decimation, to a square lattice. In spin decimation, instead of assembling blocks of a few spins each, every other spin in the lattice is held fixed while a probability distribution is computed for the remaining spins. These calculations were much more elaborate than the model calculation described here; in my own work, for example, 217 couplings between spins were included. The critical exponents derived from the calculation agree with Onsager's values to within about .2 percent.

Because an exact solution is known for the two-dimensional Ising model, the application of the renormalization group to it is something of an academic exercise. For a system of Ising spins in a three-dimensional lattice, however, no exact solution is known. A method has been devised, by Cyril Domb of University College London and many others, for finding approximate values of the exponents in the three-dimensional case. First the properties of the system at high temperature are determined with great precision, then these properties are extrapolated to the critical temperature. The best results obtained so far by this method give values for the exponents of $\beta = .33$, $\gamma = 1.25$ and $\nu = .63$.

Although extrapolation from a high-temperature solution leads to good approximations for the critical exponents, it provides little intuitive understanding of how the system behaves near the critical point. A renormalization-group calculation gives essentially the same values for the exponents, but it also explains important universal features of critical behavior.

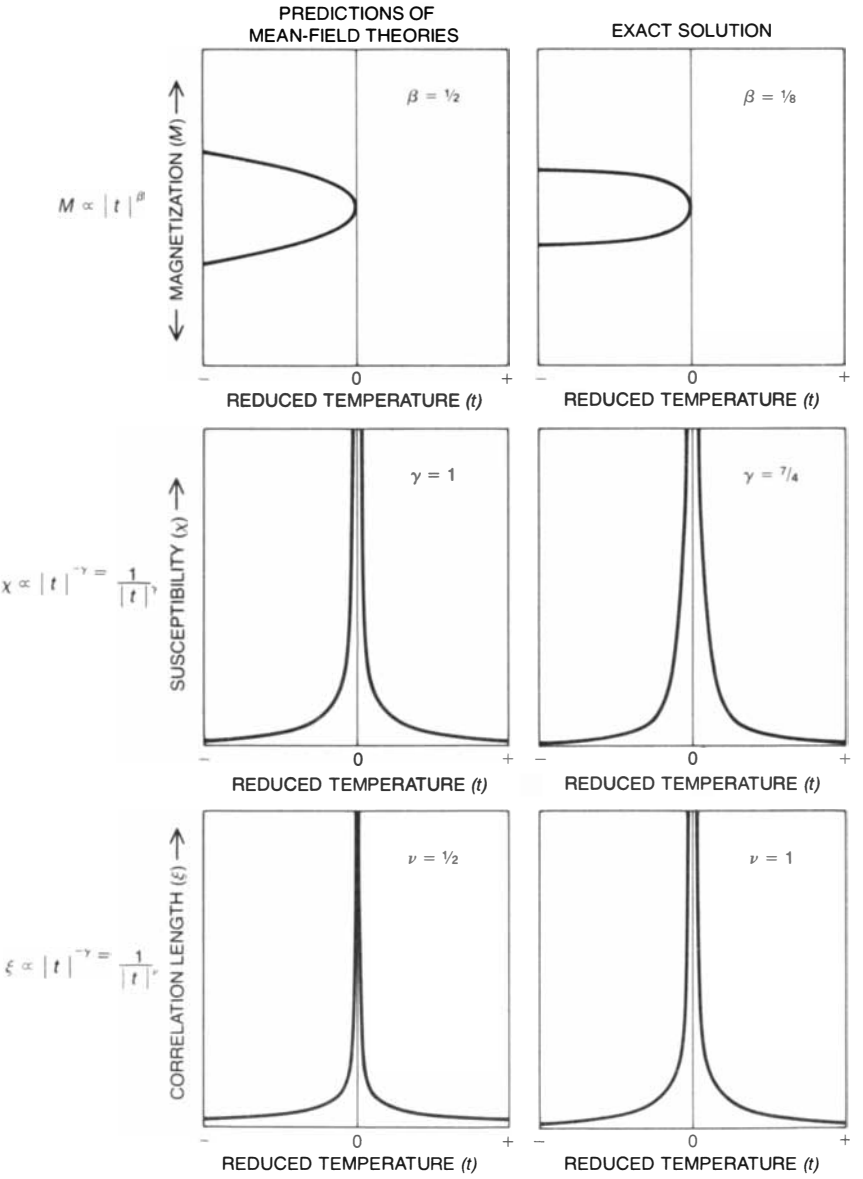
Two remarkable facts about the exponents in the three-dimensional Ising model should not be overlooked. The first is simply that the values are different from those for the two-dimensional model. In mean-field theories the dimensionality of space does not enter the calculations and so the critical exponents have the same values in any space. The second surprise is that the exponents are not integers or ratios of small integers, as they are in mean-field theories. They may even be irrational numbers.

If it is surprising that the spatial dimensionality influences the critical exponents, it is equally remarkable that certain other properties of the model have no effect at all. An example of such an irrelevant parameter is the structure of the lattice. In the two-dimensional Ising model it makes no difference whether the lattice is rectilinear, as in my own work, or triangular, as in the model employed by Niemeijer and van Leeuwen; the critical exponents are the same. By extension, in a real ferromagnet the great variety of crystal structures all yield identical critical behavior.

There is an intuitive justification for the irrelevance of the lattice structure and of other microscopic properties. A change in the form of the lattice has a large effect on events at the scale of the lattice spacing, but the effect diminishes as the scale of interest increases. In a renormalization-group calculation the fluctuations at the scale of the lattice spacing are averaged out after the first few iterations, and so models with many different lattices have the same critical behavior. Through the renormalization group the appearance of the same critical exponents in many systems is seen to result from the topography of the surface in parameter space. Each lattice structure corresponds to a different position in parameter space, but at the critical temperature every lattice is represented by a point somewhere along the ridgeline. After repeated renormalization-group transformations all these systems converge on the same fixed point, namely the saddle point.

The idea that certain variables are irrelevant to critical phenomena can be extended to systems other than ferromagnets. A fluid near its critical point, for example, has the same properties as the three-dimensional Ising model of a ferromagnet. In order for this identity to be understood some correspondence must be established between the macroscopic properties of the fluid and those of the magnet. The magnetization, which is the number of up spins minus the number of down spins, can be identified with the density difference in the fluid: the density of the liquid phase minus the density of the vapor phase. Just as the magnetization vanishes at the Curie temperature, so the density difference falls to zero at the critical point of the fluid. These quantities—the magnetization and the density difference—are called the order parameters of their respective systems. The susceptibility of the magnet, which is the change in magnetization for a given small change in the applied magnetic field, is analogous to the compressibility of the fluid: the change in density that results from a given small change in pressure. Like the susceptibility, the compressibility becomes infinite at the critical point. The critical behavior of the fluid and that of the three-dimensional Ising model are identical in that they have the same surface in parameter space. The two systems have different initial positions on the surface, but they converge on the same saddle point and hence have the same critical exponents.

The similarity observed in the critical behavior of fluids and of ferromagnets is an instance of a more general hypothesis called critical-point univer-



CRITICAL EXPONENTS express the dependence of macroscopic properties on the extent to which the temperature of the system departs from the critical temperature. The temperature is most conveniently given in the form of the reduced temperature, t, defined by the equation $t = T - T_c/T_c$. All macroscopic properties are then proportional to the absolute value of t raised to some power; the power is the critical exponent for that property. The exponents and power laws in the graphs at the left are those predicted by mean-field theories, which ignore all fluctuations. The exponents in the graphs at the right are derived from an exact solution of the two-dimensional Ising model reported in 1944 by Lars Onsager of Yale University. The exponents show how the properties of the system change as the temperature or the coupling strength is changed; that is the same information conveyed by the slope of the surface in parameter space near the critical fixed point. The exponents can be determined from the slope, and calculations by the author and others for the two-dimensional Ising model give values close to Onsager's.

sality. According to the hypothesis, only two quantities determine the critical behavior of most systems: the dimensionality of space and the dimensionality of the order parameter. These quantities are labeled respectively d and n. All systems that have the same values of d and n are thought to have the same surface in parameter space and the same critical exponents. They are said to be members of the same universality class.

The dimensionality of space is seldom difficult to determine, but the dimensionality of the order parameter requires more careful consideration. In magnetic systems, where the order parameter is the magnetization, n is the number of components needed to define the spin vector. The vector of an Ising spin can be oriented only along a single axis, and so it has only one component; for the Ising model n is equal to 1. A spin vector that is allowed to point anywhere in a plane has two components, which are customarily drawn along the two axes that define a plane. Similarly, a vector that can point anywhere in three-dimensional space has three components, so that n equals 3.

For the three-dimensional Ising model d equals 3 and n equals 1. Ordinary fluids belong to the same universality class. The space in which the fluid exists clearly has three dimensions. The order parameter—the difference in density between the liquid and the vapor phases—is a quantity that has only a magnitude and hence only one component; it can be expressed as a single number, just as the value of an Ising spin can be.

Several other physical systems are members of this class. A mixture of two liquids such as oil and water exhibits critical behavior near the temperature where the component fluids become completely miscible in each other, a temperature called the consolute point. At temperatures below the consolute point the mixture separates into two phases, and the order parameter is defined as the concentration difference between the two phases, another quantity that can be expressed as a single number. Alloys such as brass have a transition between an ordered phase, where the two metals occupy alternate sites in a regular lattice, and a disordered phase, where their distribution is less uniform. The order parameter in this system is again a concentration difference, so that n equals 1. All these systems are expected to have the same critical exponents as the three-dimensional Ising model. So are some real ferromagnets, those that are easily magnetized only along a single axis. The available experimental evidence confirms these predictions.

The universality hypothesis would be trivial if the critical exponents had the values of integers or simple fractions such as 1/2. Many physical laws share such exponents, and there is no compelling reason for postulating a connection between them. Gravitation and electromagnetism both have an inverse-square law (an exponent of -2), but that coincidence does not demonstrate that the two forces are identical. The correspondence of exponents does seem remarkable, however, when the values are not round numbers but fractions such as .63. The convergence of many systems on these values cannot be coincidental. It is evidence that all the details of physical structure distinguishing a fluid from a magnet are less important than the geometric properties expressed by the values of d and n.

The two-dimensional Ising model ($d = 2$, $n = 1$) typifies a class of systems that are confined to two-dimensional space. One example is a thin film of liquid; another is a gas adsorbed on a solid surface. An ordinary ferromagnet falls into the class with $d = 3$ and $n = 3$, that is, the lattice is three-dimensional and each spin has three components, so that it can point in any direction. When the spins are constrained to lie in a plane, the class is reduced to $d = 3$ and $n = 2$. In this same class are the superfluid transition of liquid helium 4 and the superconducting transitions of various metals.

Other universality classes have values of d and n whose interpretation is somewhat less obvious. The case of $d = 4$ is of interest in the physics of elementary particles, where one of the four spatial dimensions corresponds to the axis of time. In a theoretical lattice of spins called the spherical model, where an individual spin can have any magnitude and only the total of all the spins is constrained, n is effectively infinite. A self-avoiding random walk through a lattice of points, or in other words a random walk that never occupies the same lattice site more than once, describes the folding up in space of a long-chain polymer; Pierre Gilles de Gennes of the Collège de France has shown that this problem belongs to a universality class with n equal to zero. In theoretical models n

UNIVERSALITY CLASS		THEORETICAL MODEL	PHYSICAL SYSTEM	ORDER PARAMETER
$d = 2$	$n = 1$	Ising model in two dimensions	Adsorbed films	Surface density
	$n = 2$	XY model in two dimensions	Helium-4 films	Amplitude of superfluid phase
	$n = 3$	Heisenberg model in two dimensions		Magnetization
$d > 2$	$n = \infty$	"Spherical" model	None	
$d = 3$	$n = 0$	Self-avoiding random walk	Conformation of long-chain polymers	Density of chain ends
	$n = 1$	Ising model in three dimensions	Uniaxial ferromagnet	Magnetization
			Fluid near a critical point	Density difference between phases
			Mixture of liquids near consolute point	Concentration difference
			Alloy near order-disorder transition	Concentration difference
	$n = 2$	XY model in three dimensions	Planar ferromagnet	Magnetization
			Helium 4 near superfluid transition	Amplitude of superfluid phase
	$n = 3$	Heisenberg model in three dimensions	Isotropic ferromagnet	Magnetization
$d \leq 4$	$n = -2$		None	
	$n = 32$	Quantum chromodynamics	Quarks bound in protons, neutrons, etc.	

UNIVERSALITY HYPOTHESIS states that diverse physical systems behave identically near their critical points. In most cases the only factors that determine the critical properties are the dimensionality of space, d, and the dimensionality of the order parameter, n. For magnetic systems the order parameter is the magnetization, and its dimensionality is the number of components needed to describe the spin vector. Most systems with the same values of d and n are members of the same universality class and share the same critical exponents. For example, ferromagnets that resemble the three-dimensional Ising model, fluids, mixtures of liquids and certain alloys are all members of the class with $d = 3$ and $n = 1$; graphs of their properties near a critical point should all have the same form. The interpretation of some values of d and n is less obvious, and values such as $n = -2$ can be defined mathematically but correspond to no known physical system. The XY model and the Heisenberg model are similar to the Ising model but describe ferromagnets whose spin vectors have two and three components respectively.

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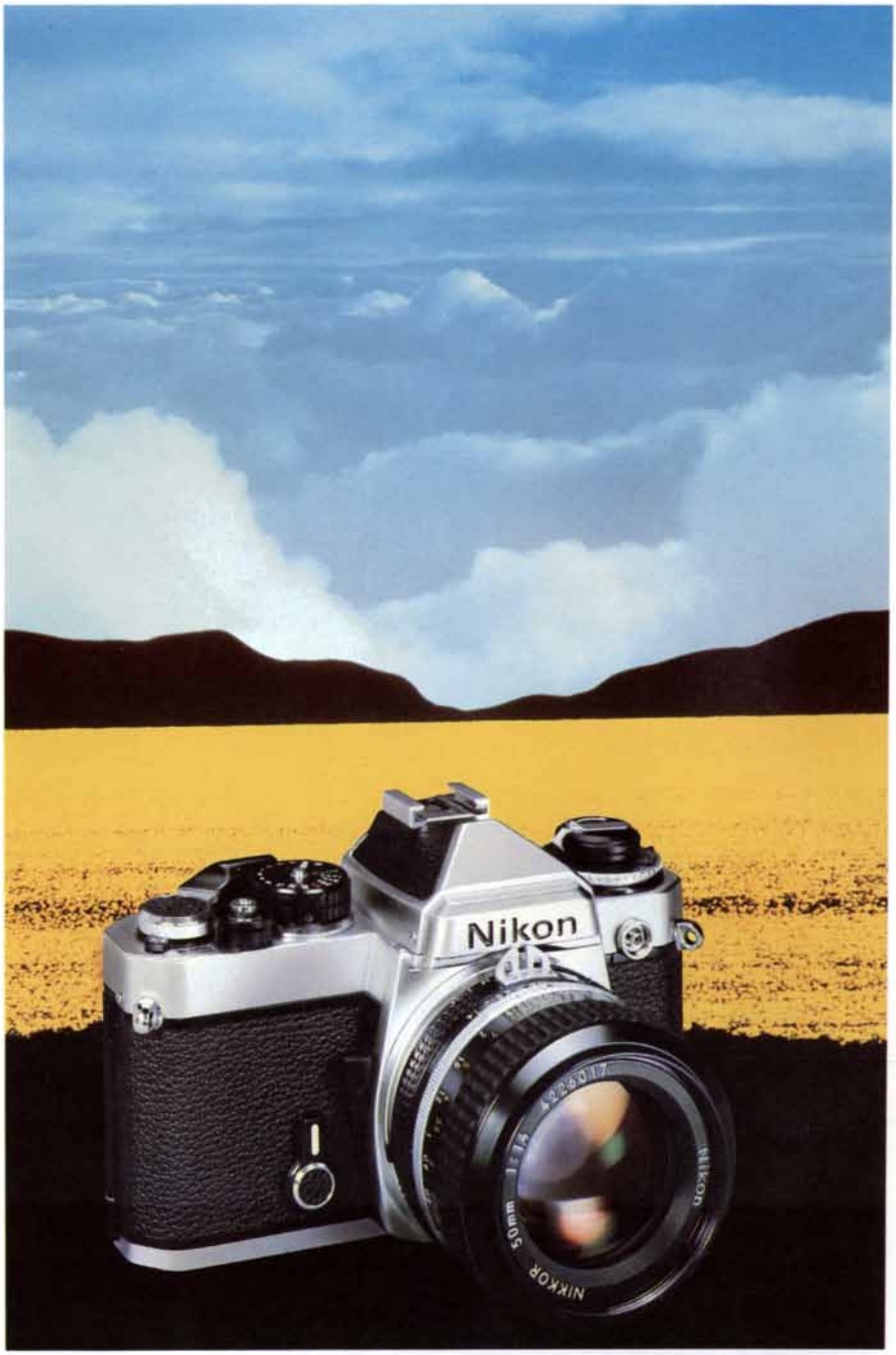

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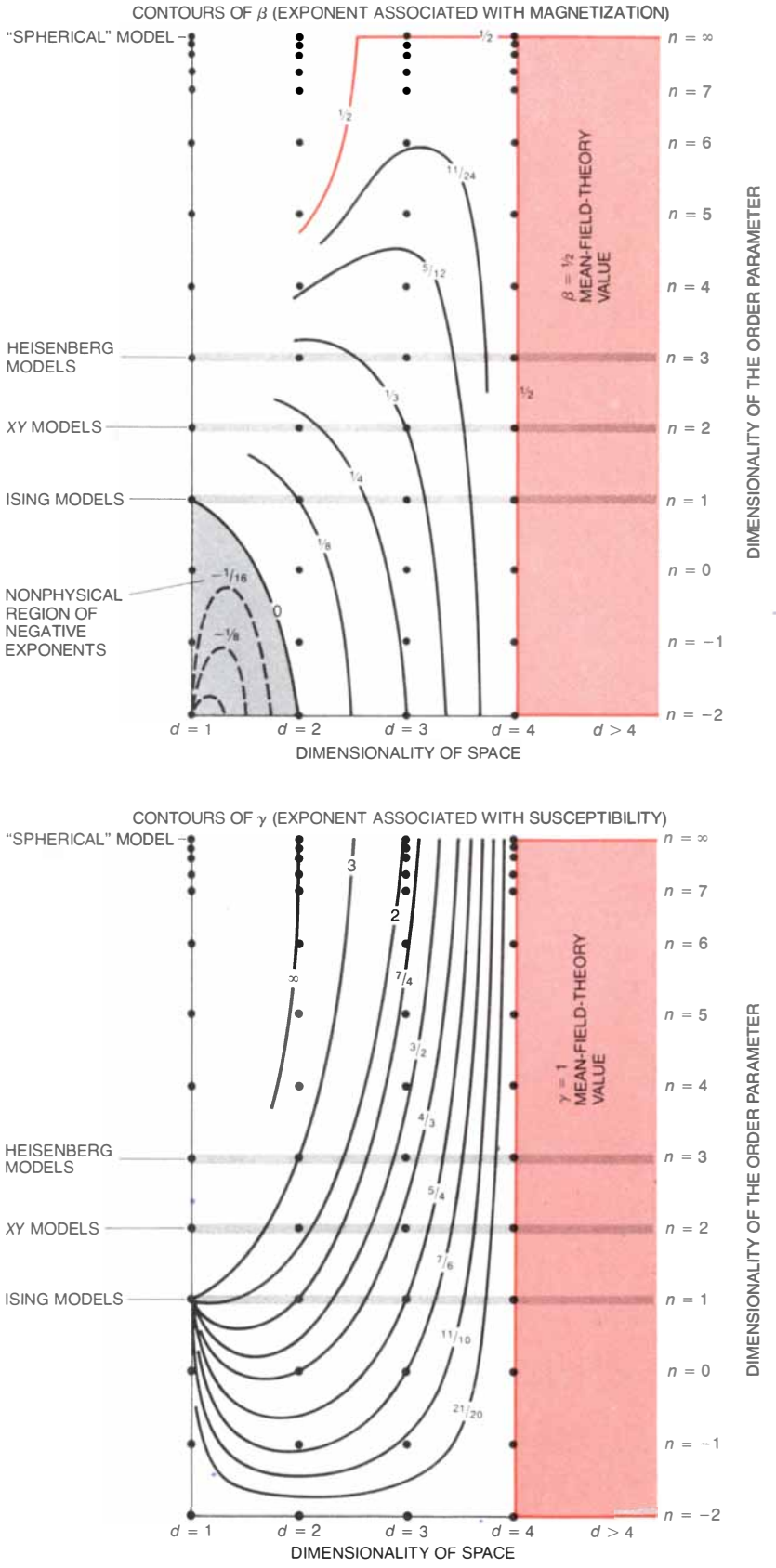
can even take on the value -2, although the physical meaning of a negative number of vector components is not clear.

The only values of d and n that have a straightforward physical meaning are integer values. This is particularly clear in the case of d, since a space with a noninteger number of dimensions is hard even to imagine. In renormalization-group calculations, however, d and n appear in equations where they can be allowed to vary continuously over some range. It is even possible to draw a graph in which the values of critical exponents are plotted as continuous functions of d and n. The exponents have well-defined values not only for integer dimensions but also for all fractional dimensions between the integers. Such a graph shows that the exponents approach the values given by mean-field theory as the number of spatial dimensions approaches 4. When d is equal to exactly 4, and at all higher values of d, the mean-field values are exact. This observation has given rise to an important method of performing renormalization-group calculations. The dimensionality of space is expressed as being equal to $4 - \epsilon$, where ϵ (the Greek letter epsilon) is a number that is assumed to be small. The critical exponents can then be calculated as the sum of an infinite series of terms including progressively higher powers of ϵ. If ϵ is less than 1, a high power of ϵ will have a small value, and reasonable accuracy can be obtained by neglecting all but the first few terms in the infinite series.

This calculation method, which is called the epsilon expansion, was developed by Michael E. Fisher of Cornell University and me. It is a general method for solving all the problems to which mean-field theory can be applied, and it represents the natural successor to Landau's theory. Indeed, it supplies answers in the form of corrections to the values given by mean-field theory. The block-spin method is the more transparent technique, but the epsilon expansion is the more powerful one.

It is not entirely surprising that the critical exponents should converge on the mean-field values as the number of spatial dimensions increases. The fundamental assumption of mean-field theories is that the force at each lattice site is influenced by conditions at many other sites. The number of nearest-neighbor sites increases along with the number of spatial dimensions. In a one-dimensional lattice each site has just two nearest neighbors, in a two-dimensional lattice four, in a three-dimensional lattice six and in a four-dimensional lattice eight. Hence as the dimensionality increases, the physical situation begins to resemble more closely the underlying hypothesis of mean-field theory. It remains a mystery, however, why $d = 4$ should mark a sharp boundary above which the mean-field exponents are exact.

In this article I have discussed main-



VARIATION OF CRITICAL EXPONENTS with the dimensionality of space (d) and of the order parameter (n) suggests that physical systems in different universality classes should have different critical properties. The exponents can be calculated as continuous functions of d and n, but only systems with an integral number of dimensions are physically possible. In a space with four or more dimensions all the critical exponents take on the values predicted by mean-field theories. The graphs were prepared by Michael E. Fisher of Cornell University.

ly the applications of the renormalization group to critical phenomena. The technique is not confined to those problems, however, and indeed it did not begin with them.

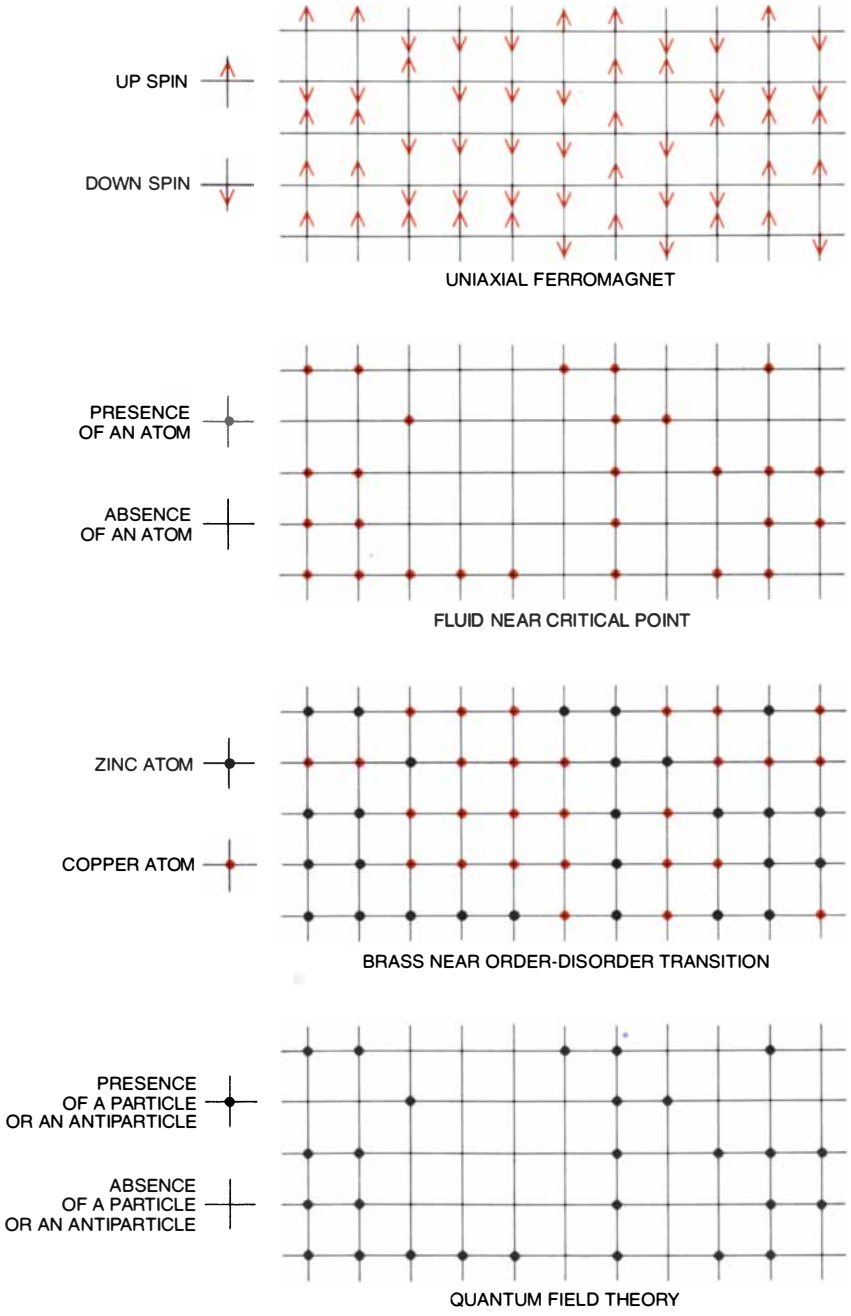
The procedure called renormalization was invented in the 1940's as part of the development of quantum electrodynamics, the modern theory of interactions between electrically charged particles and the electromagnetic field. The difficulty encountered in the formulation of the theory can be understood as one of multiple scales of length. For some time it had been apparent that the charge of the electron predicted by quantum-mechanical theories was infinite, a prediction that was in serious conflict with the measured charge. The renormalized theory of electrodynamics does not abolish the infinity; on the contrary, the electron is defined as a point particle whose "bare" charge is infinite. In quantum electrodynamics the bare charge has the effect, however, of inducing a charge of opposite polarity in the surrounding vacuum, which cancels most of the infinity, leaving only the small net charge that is observed in ordinary experiments.

One can imagine a probe particle that could measure the electron's charge at arbitrarily close range. At long range it would find the familiar finite value, which is the difference between the bare charge and the induced charge. As the layers of shielding were penetrated the measured charge would increase, and as the range was reduced to zero the charge would become infinite. The renormalization procedure provides a means for subtracting the infinite shielding charge from the infinite bare charge so that a finite difference results.

In the 1950's it was pointed out by several workers, among them Murray Gell-Mann and Francis E. Low, that the renormalization procedure adopted for quantum electrodynamics is not unique. They proposed a more general formulation, which is the original version of the renormalization group. In their application of the method to quantum electrodynamics a mathematical expression is constructed that gives the magnitude of the charge at some definite distance from the electron. Then the form of the expression is examined as the distance at which the measurement is made is allowed to approach its limiting value of zero. The arbitrariness of the procedure is in the choice of the initial distance. Any value can be selected without changing the ultimate results, so that there is an infinite set of equivalent renormalization procedures.

A "group" in mathematics is a set of transformations that meets a special requirement: the product of any two transformations must also be a member of the set. For example, rotations are transformations that make up a group, since the product of any two rotations is also a rotation. What this means in the case of the renormalization group is that the procedure can be iterated indefinitely, since applying the procedure twice is equivalent to applying the product of two transformations. Actually the renormalization group is properly called a semigroup because the inverse of the transformation is not defined. The reason for this can be seen plainly in the block-spin technique applied to the two-dimensional Ising model. A block of nine spins can be condensed into a single average spin, but the original spin configuration cannot be recovered from the average because essential information has been lost.

The version of the renormalization



LATTICE SYSTEM can be interpreted as a model not only of a ferromagnet but also of other physical systems that have fluctuations on many scales. The Ising model describes a uniaxial ferromagnet, one with a preferred axis of magnetization. It can also be applied to a fluid near its critical point, where each lattice site either is occupied by an atom or is vacant, so that the fluctuations become variations in density. An alloy such as brass has a similar structure, where each site is occupied by one kind of metal or the other. In all these systems the fluctuations are thermal; in the quantum field theories that describe the interactions of elementary particles there are quantum fluctuations of the vacuum, which allow particles and antiparticles to appear spontaneously. A simple quantum field theory can be formulated on a lattice by specifying that the particles and antiparticles can be created and annihilated only at the lattice sites.

group outlined in this article differs in several respects from the one introduced by Gell-Mann and Low. The earlier version of the technique is useful only for understanding problems that can be solved by one of the traditional methods of physics: by finding some approximate expression for the behavior of a system and then calculating better approximations as a series of perturbations departing from the original expression. Moreover, in the original formulation only one quantity is allowed to vary; in the example given above it is the charge of the electron. As a consequence the surface in parameter space is not a multidimensional landscape but a mere line. The modern version of the renormalization group, which was introduced by me in 1971, gives access to a much broader spectrum of physical problems. What is equally important, it gives a physical meaning to the renormalization procedure, which otherwise seems purely formal.

In the past few years I have been attempting to apply the newer version of the renormalization group to a problem in the physics of elementary particles. The problem is how to describe the interactions of quarks, the hypothetical elementary particles thought to compose protons, neutrons and a multitude of related particles. In one sense the problem is much like the original renormalization problem of quantum electrodynamics; in another sense it is just the opposite.

In quantum electrodynamics the charge of the electron is found to increase as the electron is approached more closely. For interactions of quarks the property analogous to electric charge is called color, and for that reason the theory of quark interactions has been named quantum chromodynamics. When the color charge of a quark is measured at close range, it seems to diminish as the distance becomes smaller. As a result two quarks that are very close together interact hardly at all: the coupling between them is weak. On the other hand, when the quarks are pulled apart, the effective color charge increases and they become tightly bound. Whereas an electron induces a compensating charge in the surrounding space, a quark seems to induce a color charge with the same polarity, which augments its own charge at long range. Indeed, it is a widely accepted hypothesis that the effective coupling between quarks increases without limit when the distance between them exceeds the diameter of a proton, which is about 10^{-13} centimeter. If that is true, a quark could be torn loose from a proton only by expending an infinite quantity of energy. The quarks would be permanently confined.

One way of visualizing the binding of quarks is to construct imaginary lines of force between them. The strength of the coupling is then proportional to the number of lines per unit area that cross any surface between the particles. In the case of electrons when the particles are separated, the lines of force spread out in space, so that there are fewer lines per unit area. The density of lines declines as the square of the separation, which yields the familiar inverse-square law for the electromagnetic force. With quarks, on the other hand, the prevailing hypothesis holds that the lines of force do not spread out in space; they remain confined to a thin tube, or string, that directly links the quarks. As a result the number of lines per unit area remains constant no matter what the distance is, and the quarks cannot be separated. Although this account of quark confinement has an intuitive appeal, it is a qualitative explanation only. No one has yet been able to derive the confinement of quarks from the underlying theory of quantum chromodynamics.

The confinement problem is one with many scales of length and energy and hence is a candidate for renormalization-group methods. I have formulated a version of the problem in which the quarks occupy the sites of a lattice in four-dimensional space-time and in which they are connected by "strings" that follow the lines connecting sites. The lattice is a strictly artificial structure with no analogue in real space-time, and it must ultimately disappear from the theory. That can be accomplished by allowing the lattice spacing to approach zero.

As in the study of ferromagnetic systems, a renormalization-group transformation is applied repeatedly to the lattice of quarks and strings. In this way the interaction of the quarks can be examined at progressively larger separations. The question to be answered is whether the lines of force remain confined to tubelike bundles or spread out in the lattice as the scale of length is increased. The calculations are near the limit of practicality for the present generation of digital computers. As yet I do not have the answers.

There are many other problems seemingly suitable for renormalization-group methods but that have not yet been expressed in such a way that they can be solved. The percolation of a fluid through a solid matrix, such as water migrating through the soil or coffee through ground coffee beans, involves aggregations of fluid on many scales. Turbulence in fluids represents a problem of notorious difficulty that has resisted more than a century of effort to describe it mathematically. It is characterized by patterns with many characteristic sizes. In the atmosphere, for example, turbulent flows range in scale from small "dust devils" to hurricanes.

One problem that has yielded to the renormalization group is a phenomenon in solid-state physics called the Kondo effect, after the Japanese physicist Jun Kondo. The effect is observed in nonmagnetic metals, such as copper, when they are contaminated with a small concentration of magnetic atoms. The simplest theories predict that the electrical resistance of such a metal will fall continuously as the temperature is reduced. Actually the resistance reaches a minimum value at a finite temperature and then rises again as the temperature is reduced further. The anomaly was never one of pressing importance because an explanation of it does not illuminate more general properties of solids, but it tantalized physicists for more than 40 years, always seeming just beyond the reach of the available methods. The root of the difficulty is that the conduction electrons in the metal can have any energy over a range of a few volts, but perturbations in that energy are significant down to a level of about 10^{-4} volt. The problem was ultimately solved in 1974, when I completed a renormalization-group calculation of the electron energies at all temperatures down to absolute zero.

A more recent series of renormalization-group calculations is notable in that it makes predictions that have been directly confirmed by experiment. The calculations concern the lattice-spin model in which d equals 2 and n equals 2, or in other words concern a two-dimensional lattice of two-component spins. It has been proved that no phase with long-range order is possible in this spin system, but renormalization-group studies done by J. M. Kosterlitz of the University of Birmingham and David J. Thouless of Yale University have shown that the behavior of the system does change abruptly at a critical temperature. These findings have been applied to studies of thin films of superfluid helium 4, which also fall into the universality class of $d = 2$ and $n = 2$. In particular Kosterlitz and David R. Nelson of Harvard University have predicted a discontinuous jump in the density of the superfluid fraction of the film. Such a jump has since been observed experimentally by John D. Reppy of Cornell and others and has been found to have the predicted magnitude.

For all the work that has been invested in the renormalization group it may seem the results obtained so far are rather scanty. It should be kept in mind that the problems to which the method is being applied are among the hardest problems known in the physical sciences. If they were not, they would have been solved by easier methods long ago. Indeed, a substantial number of the unsolved problems in physics trace their difficulty to a multiplicity of scales. The most promising path to their solution, even if it is an arduous path, is the further refinement of renormalization-group methods.