LECTURE 6: Degenerate bosons

1° Understanding chemical potential of bosons

\[ \eta(\varepsilon) = \frac{1}{\varepsilon^2 \exp(\varepsilon \mu) - 1} \] \implies \mu \leq \varepsilon_0 = 0 \text{ for } \varepsilon_0 = \frac{\hbar^2}{2m} \]

in order to keep \( \eta(\varepsilon) > 0 \)

\[ N = \sum_{\varepsilon} \frac{1}{\varepsilon^2 \exp(\varepsilon \mu) - 1} \]

\[ \iff N = \int_0^\infty \frac{D(\varepsilon) d\varepsilon}{\varepsilon \exp(\varepsilon \mu) - 1} \]

as \( T \) decreases, \( \mu \) must increase in absolute value as well to keep \( N \) constant \( \Rightarrow \) when \( \mu \) becomes zero, further decrease of \( T \) leads to loss of particles

The integral does not change if you change its value at discrete set of points of measure zero; also \( D(\varepsilon) \propto \varepsilon \) means \( D(\varepsilon) = 0 \) so any number of particles in \( \varepsilon = 0 \) is not included:

\[ N = N_0 + N_* = e^{-\mu \varepsilon_0} + g \frac{V}{4\pi^2} \frac{(2m)^{3/2}}{\hbar^3} \int_0^\infty \frac{\varepsilon \exp(\varepsilon \mu) - 1}{\exp(\varepsilon \mu) - 1} \]

canonical ensemble applies

macroscopic number of particles in Bose-Einstein condensate

only to particles above the condensate

\[ N_* = N \Rightarrow T = T_{\text{BEC}} \]

\[ N_0 = \frac{N_0}{N} = 1 - \frac{N_*}{N} \]

\[ N_0 = N \left[ 1 - \left( \frac{T}{T_{\text{BEC}}} \right)^2 \right] \]

\[ \sum_{\varepsilon} \frac{\varepsilon^{m-1}}{\exp(\varepsilon) - 1} = \Gamma(m) \zeta(m) \Rightarrow T_{\text{BEC}} = \frac{3.51 \frac{\hbar^2}{m k_B} (N)^{2/3}}{8V} \]

\[ \zeta(3/2) = 2.6124 \]
2° role of thermodynamic limit: \( N_0 = \frac{1}{e^{\mu/T} - 1} \approx \frac{k_B T}{\mu} \) for finite \( N \ll \mu \approx \frac{1}{N} \) as \( N \to \infty \frac{N}{V} = \text{const} \).

2° pressure of BEC

\[ T < T_{\text{BEC}} \Rightarrow \beta P = \frac{\partial f}{\partial \mu} = \frac{3}{2} \left( \frac{T}{T_{\text{BEC}}} \right)^{3/2} \approx 1.341 \frac{g}{\lambda^3} \neq F(n) \]

3° Is BEC of noninteracting particles first or second order phase transition?

\[ n_* = \frac{\partial^2 f}{\partial \mu^2} \left( \frac{T}{T_{\text{BEC}}} \right) \Rightarrow V^* = \frac{1}{n_*} = \frac{\lambda^3}{g \beta^{3/2}} \]

so transition can be induced by reducing volume \( V < V^* \) or increasing density, at fixed \( T \).

2° Clausius - Clapeyron relation for latent heat \( Q_l \):

\[ m_0(T, p) = m^*(T, p) \Leftrightarrow d\mu_0 = d\mu^* \Rightarrow d\mu = -S dT + V d\mu \]

\( \left( \frac{\partial \mu}{\partial T} \right)_p = -S, \quad \left( \frac{\partial \mu}{\partial p} \right)_T = V \Rightarrow \frac{\partial p}{\partial T} \left|_{\text{exist}} \right. = \frac{S^* - S_0}{V^* - V_0} = \frac{Q_l}{\Gamma_{\text{BEC}}(V - V_0)} \)
\[ \frac{dp}{dT} = \frac{5}{2} \frac{P}{T} \Rightarrow \Delta S = \frac{Q_L}{T_{BEC}} \frac{\Delta V}{V_x - V_0} \Rightarrow V_0 \equiv 0 \]

unrealistic due to no interparticle interactions

\[ Q_L = T_{BEC} \frac{dp}{dT} = \frac{5}{2} \frac{P}{T_{BEC}} \]

\[ = \frac{5}{2} \frac{g}{\alpha_3^{3/2}} \frac{\gamma}{\hbar^2} \left[ T_{BEC} \frac{\alpha_3}{\gamma} \right]^{3/2} \]

\[ \approx \frac{5}{2} \frac{\gamma^{3/2}}{\hbar^2 T_{BEC}} \approx 1.28 \frac{\gamma^{3/2}}{\hbar^2 T_{BEC}} \]

\[ \Rightarrow Q_L \neq 0 \Rightarrow \text{first order phase transition} \]

- Compressibility \( \kappa_T = \left( \frac{\partial n}{\partial p} \right)_T \)

\[ \frac{dp}{dz} = \frac{g k_B T}{\alpha_3} \frac{1}{2} f_{3/2}^+(z) \]

\[ \frac{dn}{dz} = \frac{g}{\alpha_3} \frac{1}{2} f_{1/2}^+(z) \]

\[ \text{using} \quad \frac{d}{dz} f_m^+(z) = \frac{1}{2} f_{m-1}^+(z) \quad \text{take ratio} \Rightarrow \kappa_T = f_{1/2}^+(z) \left[ n k_B T f_{3/2}^+(z) \right]^{-1} \Rightarrow \infty \quad T \to T_{BEC} \]

\[ \lim_{z \to 1} f_{1/2}^+(z) \Rightarrow \infty \]

\[ \Rightarrow \text{since} \ \kappa_T \text{diverges as} \ T \to T_{BEC} \text{ this would be continuous or second order phase transition} \]