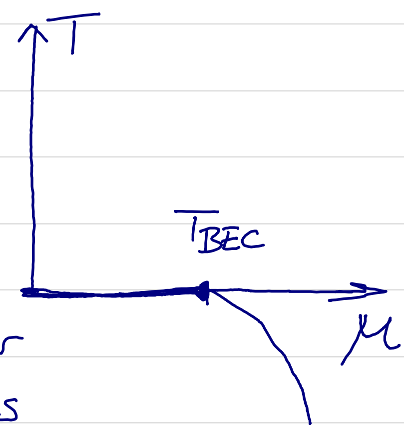
LECTURE 6: Degenerate bosons

1° Understanding chemical potential of bosons

$$n(\varepsilon) = \frac{1}{e^{\beta(\varepsilon-\mu)}-1} \Rightarrow \mu \leq \varepsilon_0 = 0 \text{ for } \varepsilon_{\vec{k}} = \frac{\hbar^2 k^2}{2m}$$

in order to keep $n(\varepsilon) > 0$

$$N = \sum_{\vec{k}} \frac{1}{e^{\beta(\varepsilon_{\vec{k}}-\mu)}-1} \overset{?}{\Longleftrightarrow} N = \int_0^\infty \frac{D(\varepsilon)d\varepsilon}{e^{\beta(\varepsilon-\mu)}-1}$$



→ as T decreases, μ MUST desrease in absolute value as well to keep N constant ⇒ when μ becomes zero, further dicrease of T leads to loss of particles

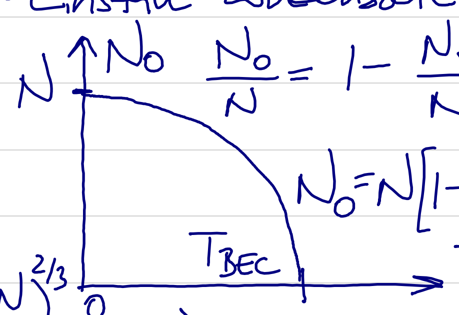
→ integral does not change if you change its value at discrete set of points of measure zero; also $D(\varepsilon) \propto \sqrt{\varepsilon}$ means $D(\varepsilon)=0$ so any number of particles in $\varepsilon=0$ is not included:

$$N = N_0 + N_* = \underbrace{\frac{g}{e^{-\beta\mu}-1}} + g\frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int_0^\infty \frac{\sqrt{\varepsilon}\,d\varepsilon}{e^{\beta(\varepsilon-\mu)}-1}$$

grand canonical ensemble applies only to particles above the condensate

macroscopic number of particles in Bose-Einstein condensate

$$\boxed{N_* = N \Rightarrow T = T_{BEC}}$$



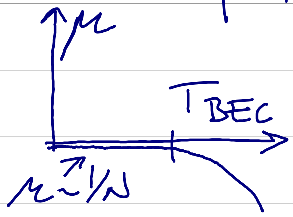
$$\frac{N_0}{N} = 1 - \frac{N_*}{N}$$

$$N_0 = N\left[1 - \left(\frac{T}{T_{BEC}}\right)^{3/2}\right]$$

$$\frac{N}{V} = g\,\frac{(2mk_BT_{BEC})^{3/2}}{4\pi^2\hbar^3}\int_0^\infty \frac{x^{1/2}dx}{e^x-1}$$

$$\int_0^\infty \frac{x^{m-1}dx}{e^x-1} = \Gamma(m)\zeta(m) \Rightarrow T_{BEC} = \frac{3.31\hbar^2}{mk_B}\left(\frac{N}{gV}\right)^{2/3} \Leftarrow \zeta(3/2) = 2.6124$$

→ role of thermodynamic limit: $N_0 = \frac{1}{e^{-\beta\mu}-1} \approx \frac{-k_B T}{\mu}$



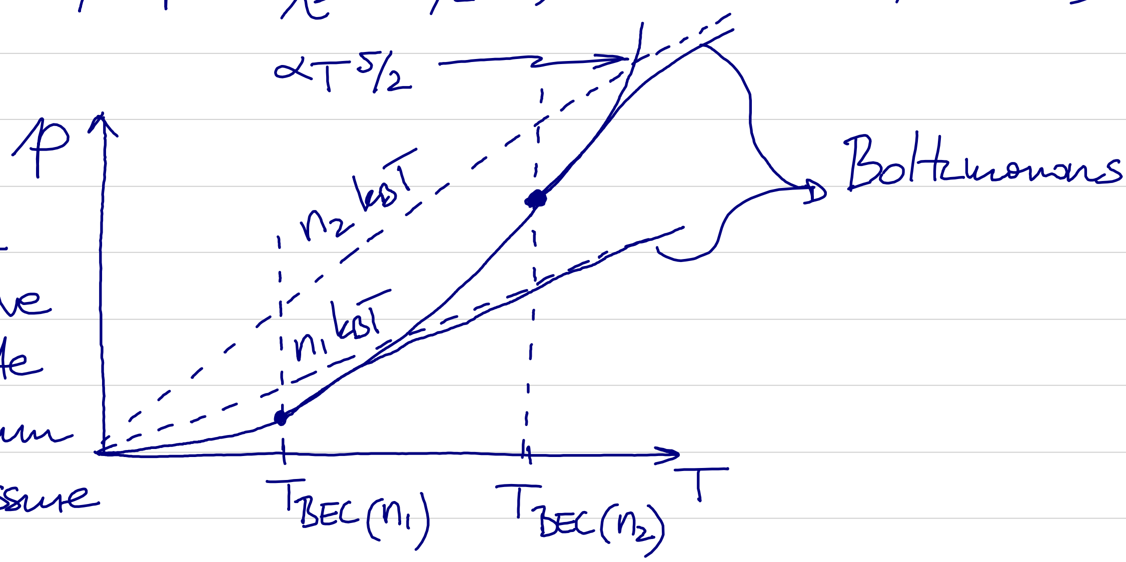
$\Big\}$ for finite N $\Leftarrow$ $\mu \sim \frac{1}{N}$ as $\begin{matrix} N\to\infty \\ V\to\infty \end{matrix}$ $\frac{N}{V} = \text{const.}$

2° pressure of BEC

$z = e^{\beta\mu} = 1$ in the condensate

$$T < T_{BEC} \Rightarrow \beta p = \frac{g}{\lambda^3} f^+_{5/2}(1) \approx 1.341\, g/\lambda^3 \neq F(n)$$

→ only N_* particles above the condensate have momentum to exert pressure



3° Is BEC of noninteracting particles first or second order phase transition?

$$n_* = \frac{g}{\lambda^3} f^+_{3/2}(1) \Rightarrow v^* = \frac{1}{n_*} = \frac{\lambda^3}{g\,\zeta(3/2)}$$

so transition can be induced by reducing volume $v < v_*$ or increasing density at fixed T

→ Clausius – Clapeyron relation for latent heat Q_L:

$$\mu_0(T,p) = \mu_*(T,p) \Leftrightarrow d\mu_0 = d\mu_* \Rightarrow d\mu_\alpha = -S_\alpha dT + V_\alpha dp$$

$$\left(\frac{\partial\mu}{\partial T}\right)_p = -S, \quad \left(\frac{\partial\mu}{\partial p}\right)_T = V \Rightarrow \frac{\partial p}{\partial T}\Big|_{coexist} = \frac{S_* - S_0}{V_* - V_0} = \frac{Q_L}{T_{BEC}(V_* - V_0)}$$

$$dp/dT = \frac{5}{2}\, p/T_{BEC} = \frac{\Delta S}{\Delta V} = \frac{Q_L}{T_{BEC}(V_* - V_0)}$$

$\to V_0 \equiv 0$ unrealistic due to no interparticle interactions

$$Q_L = T_{BEC} V_* \frac{dp}{dT} = T_{BEC} V_* \frac{5}{2} \frac{p}{T_{BEC}} =$$

$$= \frac{5}{2} \frac{g}{\lambda^3} \zeta\left(\frac{5}{2}\right) k_B T_{BEC} \left[\frac{g}{\lambda^3} \zeta(3/2)\right]^{-1} = \frac{5}{2} \frac{\zeta(5/2)}{\zeta(3/2)} k_B T_{BEC} \approx 1{,}28\, k_B T_{BEC}$$

$\Rightarrow Q_L \neq 0 \Rightarrow$ first order phase transition

- compressibility $\varkappa_T = \left(\frac{\partial n}{\partial p}\right)_T / n$

$$\left.\begin{aligned} dp/dz &= \frac{g k_B T}{\lambda^3} \frac{1}{z} f^+_{3/2}(z) \\ dn/dz &= \frac{g}{\lambda^3} \frac{1}{z} f^+_{1/2}(z) \end{aligned}\right\} \text{ using } \frac{d}{dz} f^+_m(z) = \frac{1}{z} f^+_{m-1}(z)$$

take ratio $\Rightarrow \varkappa_T = f^+_{1/2}(z) \left[n k_B T f^+_{3/2}(z)\right]^{-1} \xrightarrow[T \to T_{BEC}]{} \infty$

$$\lim_{z \to 1} f^+_{1/2}(z) \to \infty$$

$\to$ since $\varkappa_T$ diverges as $T \to T_{BEC}$ this would be continuous or second order phase transition

- heat capacity $C_V = \left(\frac{\partial E}{\partial T}\right)_{V,N}$

$$E = \frac{3}{2} p \cdot V = \frac{3}{2} V \frac{g}{\lambda^3} k_B T f^+_{5/2}(z) \propto T^{5/2} f^+_{5/2}(z)$$

$$C_V = \frac{3}{2} V \frac{g}{\lambda^3} k_B T \left[\frac{5}{2T} f^+_{5/2}(z) + \frac{1}{z} f^+_{3/2}(z) \left.\frac{\partial z}{\partial T}\right|_{V,N}\right]$$

N is fixed $\Rightarrow dN/dT|_V = \frac{g}{\lambda^3} V \left[\frac{3}{2T} f^+_{3/2}(z) + \frac{1}{z} f^+_{1/2}(z) \left.\frac{dz}{dT}\right|_{V,N}\right]$

$$\frac{T}{z}\frac{dz}{dT}\bigg|_{V,N} = -\frac{3}{2}\frac{f^+_{3/2}(z)}{f^+_{1/2}(z)}$$

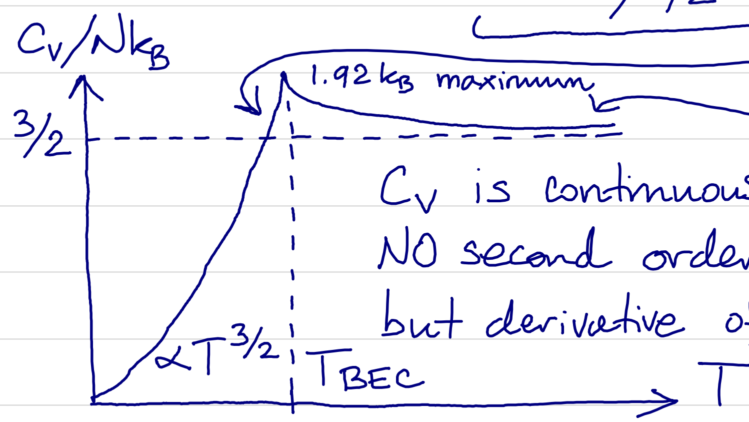
$$\frac{C_V}{V k_B} = \frac{3}{2}\frac{g}{\lambda^3}\left[\frac{5}{2} f^+_{5/2}(z) - \frac{3}{2}\frac{f^+_{3/2}(z)}{f^+_{1/2}(z)}\right]$$

at high T

$$\approx \frac{3}{2}\Big[1 + \underbrace{n\lambda^3/2^{7/2}}_{\text{larger than the classical value}} + \dots\Big]$$

$\rightarrow$ at low T $z = e^{\beta\mu} = 1$:

$$\frac{C_V}{N k_B} = \frac{15}{4}\frac{g}{n\lambda^3}\zeta_{5/2} = \underbrace{\frac{15}{4}\frac{\zeta_{5/2}}{\zeta_{3/2}}\left(\frac{T}{T_{BEC}}\right)^{3/2}}$$



C_V is continuous at T_{BEC}, so NO second order phase transition but derivative of C_V has jump at T_{BEC}

$\rightarrow$ INTUITIVE interpretation of $C_V \propto T^{3/2}$:

- at $T = 0\,K$ all particles are in state with $\vec{k} = 0$
- at T slightly above $0\,K$, particles occupy states with $\leq k_{max}$ where $\hbar^2 k_{max}^2/2m = k_B T$ and each of these states has energy $\propto k_B T$
- $E \propto V k_{max}^d \, k_B T$ for $V k_{max}^d$ as the density of states in d-dimensions $\Rightarrow C_V \propto V k_B T^{d/2}$