Light Emitted by Stars: Why is the Radiation from the Sun so Stable?

STELOAR ASTROPHYSICS: four fundamental forces of nature come into play in a characteristic and spectacular manner → the stars are formed by a collapse of matter caused by gravitational attraction; the light that they emit is generated by electromagnetic interactions; strong interactions provide their main source of energy; and weak interactions contribute in a crucial way to make their lifetime so long.

Sun as a main sequence star:

It is useful to keep in mind various orders of magnitude corresponding to the three principal classes of stars. Our Sun is a typical example of the so-called main sequence. Its radius is $R_\odot = 7 \times 10^8$ m, its mass is $M_\odot = 2 \times 10^{30}$ kg, its luminosity is $L_\odot = 3.8 \times 10^{26}$ W, and its surface temperature is $T_{s,\odot} = 6000$ K. Hence, its average density is $1.4 \text{ g cm}^{-3}$, comparable to that of water on earth. The masses of all the stars lie between $0.1 M_\odot$ and $100 M_\odot$. The Sun is mainly made of hydrogen, with 28% of its mass consisting of $^4\text{He}$ nuclei and 2% of other light elements. Its number of protons is the order of $10^{57}$. Heavier stars include red giants, a branch detached from the main sequence.

1937 Bethe and von Weizsäcker: nuclear fusion of hydrogen into helium that take place in the central part of the Sun produce some amount of heat per unit time, which is exactly equal to the luminosity because the state of the Sun is stationary. However, such reactions are very sensitive to small changes in the temperature: they are activated by a rise, hindered by a decrease. Thus, if it happens at some instant that a little more power is produced in the core than what is evacuated by radiation from the surface, why does the internal temperature not rise, eventually resulting in an explosion of the Sun? Conversely if the opposite perturbation occurs, why does the Sun not become extinct?

$$\frac{dE}{dt} = Q - L$$

stability of stellar equilibrium ensured by gravitational force
Hertzsprung-Russell Diagram and Evolution of Solar-Mass Stars
Nucleosynthesis and Fusion Reactions inside Main Sequence Stars

- In both proton-proton chain and CNO cycle, Coulomb repulsion must be overcome to initiate fusion, which involves quantum tunneling and requires extremely high kinetic energy of fusing particles.

- The CNO cycle becomes the chief source of energy in stars of 1.5 solar masses or higher.
Internal Structure of Post-Main Sequence Stars in Pictures

The Triple Alpha Process (Helium Fusion)

Hydrogen Shell Burning on the Red Giant Branch

Close-up of core region for a 1 $M_\odot$ Asymptotic Giant Branch star

Helium layer

Helium-burning shell

Carbon-oxygen core (no fusion)

(not to scale)

Internal shell structure of a supergiant on its last day

Core region

Fe core

Supergiant star

H shell

He shell

C, O shell

Si, S shell

(not to scale)
Pressure due to Classical Thermal Motion Acts Against Gravity Collapse in Main Sequence Stars

\[ U \approx -\frac{GM^2}{R} \]

star wants to contract to a state of lower energy (i.e., larger negative values of \( U \)), unless there is outward direct pressure to resist the contraction

\[ PV = Nk_B T \]
in ordinary stars with fuel for thermonuclear fusion, outward pressure is provided by thermal motion

\[
\frac{\rho GM}{2R} \approx \frac{Nmk_B T}{V_m} = \frac{\rho k_B T}{m}
\]
gravitation pressure at the center of the star is equal to thermal pressure

\[
\frac{GM^2}{R} \approx -U \approx Nk_B T
\]
total thermal energy is comparable to the magnitude of total gravitational potential energy
White Dwarfs Stabilized by Fermi Pressure of Non-Relativistic Electrons

\[ A, Z, A-Z, x = Z/A \]
\[ M \approx Nm_p, N_e = xN \]
\[ \varepsilon_F = \frac{p_F^2}{2m_e} = \frac{h^2}{2m_e} \left( \frac{3N_e}{8\pi V} \right)^{2/3} \]
\[ E_e = \frac{3}{5} N_e \varepsilon_F \]

Fermi energy of electron gas in non-relativistic approximation

Total kinetic energy of electron gas

\[ E(R) = \frac{3}{5} N_e \varepsilon_F - \frac{3GN^2m_p^2}{5R} \]

Total energy of cool star where thermal energy can be neglected

\[ \frac{d}{dR} E(R) = 0 \Rightarrow R = \frac{xh^2}{4m_e} \left( \frac{9}{4\pi^2} xN \right)^{2/3} \frac{1}{GNM_p^2} \]

White dwarfs cool off and contract to this radius

\[ M = 0.85M_\odot, N = 10^{57}, x = \frac{1}{2} \Rightarrow R \approx 8000 \text{ km}, \rho = 3 \times 10^6 \text{ g/cm}^3 \]
White Dwarfs Stabilized by Fermi Pressure of Relativistic Electrons

\[ p_F = \hbar \left( \frac{3N}{8\pi V} \right)^{1/3} \]

\[ \varepsilon_F = \sqrt{c^2 p_F^2 + m_e^2 c^4} - m_e c^2 \quad p_F \gg m_e c \]

energy per electron increases with N, so when it becomes comparable to \( mc^2 \) we have to switch to relativistic energy-momentum dispersion

\[ E(R) = \frac{3}{8} \frac{xN\hbar c}{R} \left( \frac{9}{4\pi^2} xN \right)^{1/3} - \frac{3GN^2 m_p^2}{5R} \]

for simplicity we assume uniform density assumed, while in reality density is larger in the center of the star than further out

both terms depend on R, so total energy of star decreases continuously with decreasing radius

\[ E(R) = 0 \Leftrightarrow \frac{3}{8} \frac{xN_c \hbar c}{R} \left( \frac{9}{4\pi^2} xN_c \right)^{1/3} = \frac{3GN_c^2 m_p^2}{5R} \Rightarrow N_c = \frac{3}{16} \left( \frac{125\pi}{x^2} \right)^{1/2} \left( \frac{\hbar c}{2\pi Gm_p^2} \right)^{3/2} \]

\[ N_0 = 2.2 \cdot 10^{57} \]

numerically exact result called “Chandrasekhar limiting mass” as the largest mass a white dwarf can have and still cool off to a stable cold state with finite radius and density

\[ N_c = 0.7 \left( \frac{x}{0.5} \right)^2 N_0 \Rightarrow N_c m_p = 1.4M_\odot \]
Neutron Stars Stabilized by Fermi Pressure of Neutrons

\[ e^- + p \rightarrow n + \nu \]

\[ m_p \left( \frac{8\pi}{3x} \right) \left( \frac{m_e c}{h} \right)^3 = \frac{0.97}{x} \times 10^6 \text{ g/cm}^3 \ll 10^{11} \text{ g/cm}^3 < \rho < m_p \left( \frac{8\pi}{3(1-x)} \right) \left( \frac{m_n c}{h} \right)^3 = \frac{6}{1-x} \times 10^{15} \text{ g/cm}^3 \]

E(R) could be lowered still further by changing \( x \) through inverse beta decay which requires electrons with large kinetic energy.

\[ \nu \rightarrow e^- + p \]

density of star at which Fermi momentum of electrons equals \( m_e c \)

neutrons become more abundant than protons at these densities

\[ E(R, x) = E_{\text{neutron}} + E_{\text{electron}} + E_{\text{gravitation}} \]

\[ = \frac{3}{5} \left( 1-x \right) N h^2 \left( \frac{9}{4\pi^2} \right) (1-x) N \right)^{2/3} + \frac{3}{8} x N h c \left( \frac{9}{4\pi^2} x N \right)^{1/3} \]

neglects energy of protons (since they are small fraction of nucleons in neutron star) and nuclear forces

\[ \partial_x E(R, x) = 0; \partial_R E(R, x) = 0 \Rightarrow \]

\[ x = 0 \Rightarrow R = \frac{h^2}{4m_p} \left( \frac{9}{4\pi^2} N \right)^{2/3} \left( GN m_p^2 \right)^{-1} \]

neutrons star radius is about 1000 times smaller than the white dwarf radius

\[ M = M_\odot; R = 12.6 \text{ km}; \rho = 2.4 \times 10^{14} \text{ g/cm}^3 \Rightarrow x = 0.5\% \]

since it is composed of 1-x=99.5% of neutrons, such star is effectively a giant atomic nucleus

PHYS813: Quantum Statistical Mechanics

QSM for Stellar Astrophysics
Observing Pulsar-Type Neutron Stars Through Their Radiation

NASA Chandra satellite imaging of rings created by the X-rays from a Circinus-X1 double star system, containing neutron star orbiting around another massive star, reflecting off different dust clouds.
At the density and pressure at which a neutron-quark phase transition is expected to occur, the system of neutrons is not ideal degenerate Fermi gas.

\[ n \rightarrow 2d + u \quad \text{neutron density} > 10 \times 0.15 \text{ fm}^{-3} \]

\[ \mu_n = 2 \mu_d + \mu_u \quad \text{chemical equilibrium} \]

\[ \frac{2}{3} n_u - \frac{1}{3} n_d = 0 \Rightarrow n_u = \frac{1}{2} n_d \quad \text{charge conserved} \]

\[ P_n = P_d + P_u \quad \text{pressure equilibrium} \]

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**Table 1: Theoretical properties of strange quark stars and neutron stars compared.**

<table>
<thead>
<tr>
<th>Strange Quark Stars</th>
<th>Neutron Stars</th>
</tr>
</thead>
<tbody>
<tr>
<td>Made entirely of deconfined up, down, strange quarks, and electrons</td>
<td>Nucleons, hyperons, boson condensates, deconfined quarks, electrons, and muons</td>
</tr>
<tr>
<td>Absent</td>
<td>Superfluid neutrons</td>
</tr>
<tr>
<td>Absent</td>
<td>Superconducting protons</td>
</tr>
<tr>
<td>Color superconducting quarks</td>
<td>Color superconducting quarks</td>
</tr>
<tr>
<td>Energy per baryon ( \lesssim 930 \text{ MeV} )</td>
<td>Energy per baryon ( &gt; 930 \text{ MeV} )</td>
</tr>
<tr>
<td>Maximum mass ( \sim 2 M_\odot )</td>
<td>Bound by gravity</td>
</tr>
<tr>
<td>No minimum mass if bare</td>
<td>Same</td>
</tr>
<tr>
<td>Radii ( R \lesssim 10 - 12 \text{ km} )</td>
<td>Minimum mass ( \sim 0.1 M_\odot )</td>
</tr>
<tr>
<td>Baryon numbers ( B \lesssim 10^{67} )</td>
<td>Radii ( R \gtrsim 10 - 12 \text{ km} )</td>
</tr>
<tr>
<td>Electric surface fields ( \sim 10^{15} \text{ V/cm} )</td>
<td>Baryon numbers ( 10^{66} \lesssim B \lesssim 10^{67} )</td>
</tr>
<tr>
<td>Can either be bare or enveloped in thin nuclear crusts (masses ( \lesssim 10^{-5} M_\odot ))</td>
<td>Absent</td>
</tr>
<tr>
<td>Maximum density of crust set by neutron drip, i.e., strange stars posses only outer crusts</td>
<td>Always have nuclear crusts</td>
</tr>
<tr>
<td>Does not apply, i.e., neutron stars posses inner and outer crusts</td>
<td>Form two-parameter stellar sequences</td>
</tr>
<tr>
<td>Form one-parameter stellar sequences</td>
<td></td>
</tr>
</tbody>
</table>
Insufficient Pressure → Black Holes and Their Thermodynamics

\[
\left( \varepsilon - G \frac{\varepsilon}{c^2 M} \frac{1}{R_0} \right) \leq 0
\]

Photon of energy \( \varepsilon \) cannot escape the surface of the star to reach an observer \( c \)

\[ R_0 \approx 3 \text{ km for } M = M_\odot \]

\[ R_0 = \frac{2GM}{c^2} \]

Schwarzschild radius

Factor 2 comes from general relativity which must be used when gravitation potential energy is comparable to other energies

\[ S = \frac{k_B c^3}{4 \pi \hbar} A \]

Bekenstein-Hawking entropy of a black hole

\[
\frac{1}{T} = \frac{\partial S}{\partial E} = \frac{1}{c^2} \frac{\partial}{\partial M} \left[ \frac{k_B c^3}{4 \pi \hbar} 4 \pi \left( \frac{2GM}{c^2} \right)^2 \right] = \frac{8\pi k_B G}{\hbar c^3} M
\]

\[ T = \frac{M_\odot}{M} 6.169 \times 10^{-8} \text{ K} \]

HAWKING RADIATION: Particle-antiparticle pairs created from vacuum normally quickly annihilate each other, but near the horizon of a black hole, it's possible for one to fall in before the annihilation can happen, in which case the other one escapes as Hawking radiation

\[
\frac{1}{A} \frac{\partial E}{\partial t} = -\sigma T^4 \Rightarrow \frac{d}{dt} \left( Mc^2 \right) = -\frac{\pi^2 k_B^4}{60 \hbar^3 c^2} \left( \frac{16\pi G^2}{c^4} \right) \left( \frac{\hbar c^3}{8\pi k_B G M} \right)^4
\]

Hawking radiation evaporates black hole much longer than the age of the Universe \( \sim 10^{18} \)

\[
M^2 \frac{dM}{dt} = -\frac{\hbar c^4}{15360 G^2} \equiv -b \Rightarrow M(t) = (M_\odot^3 - 3bt)^{1/3} \rightarrow 0 \text{ after } \tau \approx 2.2 \times 10^{74} \text{ s} \]