Tutorial on Renormalization Group
Applied to Classical and Quantum Critical Phenomena

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The Task for Renormalization Group (RG) Theory

RG was developed to understand:
- why divergences occur in quantum field theory (i.e., why parameters like the electron mass are scale dependent).
- universality and scaling in second-order or continuous phase transitions

K. G. Wilson
“Problems in physics with many scales of length,” Scientific American 241(2), 140 (1979)
Quick and Dirty Introduction Using Numerical Renormalization Group (RG) for Ising Model

\[ T < T_c \quad T \approx T_c \quad T > T_c \]

Block spin transformation leads to RG flow

The RG flow changes the effective temperature, where the critical point corresponds to the unstable point of the flow.

Result of RG procedure:
A renormalized system which has the same long-wavelength properties as the original system, but fewer degrees of freedom.
**RG Steps in Pictures**

**Real-Space**

**Original**

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**Coarse-grained**

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**Rescaled**

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**Momentum-Space**

**CONVENTIONS**

\[
H = -\frac{J}{k_B T} \sum_{\langle ij \rangle} S_i S_j = -K \sum_{\langle ij \rangle} S_i S_j \\
Z = \sum_{\{S_i\}} e^{-H(\{S_i\})}
\]

**VOCABULARY**

- coarse-graining
- RG recursion relations
- RG is semi-group
- fixed point definition
- rescaling

\[
Z_N(H) = \sum_{\{S_i\}} \sum_{\{S_i'\}} e^{-H(\{S_i, S_i'\})} = \sum_{\{S_i\}} e^{-H(\{S_i\})} = Z_N'(H')
\]

\[
H' = R_b(H)
\]

\[
H'' = R_{b_2}(H') = R_{b_2} \cdot R_{b_1}(H) = R_{b_1b_2}(H)
\]

\[
H^* = R_b(H^*) = \lim_{n \to \infty} R^n_b(H_c)
\]

\[
r' = b^{-1} r, \quad N' = b^{-d} N, \quad f(H') = b^d f(H)
\]
Real-Space RG for 1D Ising Model

\[ K_{20} < 10^{-6}; \quad K \rightarrow \begin{cases} \infty, \text{ferromagnet} \\ 0, \text{paramagnet} \end{cases} \]

Teaching the renormalization group

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(Received 24 August 1977; accepted 29 October 1977)

The renormalization group theory of second-order phase transitions is described in a form suitable for presentation as part of an undergraduate statistical physics course.
RG Flow for 1D Ising Model

\[ Z_N(K) = \sum_{\{S_i\}} e^{K(S_1 S_2 + S_2 S_3 + \ldots)} = \sum_{\{S_i\}} \left[ e^{K S_1 S_2} e^{K S_2 S_3} \right] \left[ e^{K S_3 S_4} e^{K S_4 S_5} \right] \ldots \]

\[ = \sum_{\{S_i \neq S_2\}} \left[ e^{K S_1 + K S_3} + e^{-K S_1 - K S_3} \right] \left[ e^{K S_3 S_4} e^{K S_4 S_5} \right] \ldots \]

\[ e^{K S_1 + K S_3} + e^{-K S_1 - K S_3} = A(K) e^{B(K) S_1 S_3} \]

\[ S_1 = S_3 = \pm 1 \Rightarrow 2 \cosh(2K) = A(K) e^{B(K)} \]

\[ S_1 = -S_3 = \pm 1 \Rightarrow 2 = A(K) e^{-B(K)} \]

\[ A(K) = 2 \sqrt{\cosh(2K)}; \quad B(K) = \frac{1}{2} \ln \cosh(2K) = K_1 \]

\[ Z_N(K) = \left[ A(K) \right]^{N/2} Z_{N/2}(K_1) \]

\[ K_{i+1} = \frac{1}{2} \ln \cosh(2K_i) \]

\[ K^* = \frac{1}{2} \ln \cosh(2K^*) \]

RG flow does not contain non-trivial fixed points, which means no phase transition at any finite temperature

stable fixed point \( 0 \)

unstable fixed point \( \infty \)
Real-Space RG for 2D Ising Model

\[ e^{K(S_1+S_2+S_3+S_4)} + e^{-K(S_1+S_2+S_3+S_4)} = A(K)e^{B(K)[S_1S_2+S_2S_3+S_3S_4+S_4S_1+S_1S_3+S_2S_4]} + C(K)S_1S_2S_3S_4 \]

\[ S_1 = S_2 = S_3 = S_4 = \pm 1 \Rightarrow 2 \cosh 4K = A(K)e^{6B(K)+C(K)} \]

\[ S_1 = S_2 = S_3 = -S_4 = \pm 1 \Rightarrow 2 \cosh 2K = A(K)e^{-C(K)} \]

\[ S_1 = S_2 = -S_3 = -S_4 = \pm 1 \Rightarrow 2 = A(K)e^{-2B(K)+C(K)} \]

\[ A(K) = 2\sqrt{\cosh 2K} \sqrt[8]{\cosh 4K} ; \quad B(K) = \frac{1}{8} \ln \cosh 4K ; \quad C(K) = \frac{1}{8} \ln \cosh 4K - \frac{1}{2} \ln \cosh 2K \]

\[ Z_N(K) = \left[ A(K) \right]^{N/2} Z_{N/2}(K_1, K_2, K_3) \]

\[ H = -2B(K)\sum_{\langle ij \rangle} S_iS_j - B(K)\sum_{[ij]} S_iS_j - C(K) \sum \text{square } S_iS_jS_kS_l \]

If we set \( K_2=K_3=0 \), we get similar trivial RG flow as in 1D Ising model:

\[ K_{i+1}^1 = \frac{1}{4} \ln \cosh 4K_i^1 \]
RG Flow for 2D Ising Model

0 [a posteriori we indeed find \( K'_3(K_c) = -0.05323 \)]

Both \( K_1 \) and \( K_2 \) are positive and thus favor the alignment of spins. Hence, we will omit the explicit presence of \( K_2 \) in the renormalized energy, but increase \( K_1 \) accordingly to a new value \( K' = K'_1 + K'_2 \) so that the tendency toward alignment is approximately the same.

\[
K'_1 = \frac{1}{4} \ln(\cosh 4K), \\
K'_2 = \frac{1}{8} \ln(\cosh 4K), \\
K'_3 = -\frac{1}{8} \ln(\cosh 4K) - \frac{1}{2} \ln(\cosh 2K)
\]
Scaling Hypothesis and Critical Exponents
Extracted from RG for 2D Ising Model

\[ C_V = -T \left( \frac{\partial^2 F}{\partial T^2} \right)_V \]
\[ C_V \sim |T - T_c|^{-\alpha} \sim |t|^{-\alpha} \]

\[ \ln Z = N f(K) \]
\[ f(K') = 2f(K) - \ln \left[ 2(\cosh 4K)^{1/8}(\cosh 2K)^{1/2} \right] \]

removal of short-range degrees of freedom results in an expression that is analytic in \( K \)

\[ f_s(K) = b^{-d} f_s(K') ; \quad b = \sqrt{2} \text{ and } d = 2 \]

focus on the singular part of free energy

\[ K' = R(K) = K_c + \frac{dR}{dK} \bigg|_{K=K_c} (K - K_c) + \cdots \]
\[ \lambda = R'|_{K_c} ; \quad \lambda(b) \lambda(b) = \lambda(b^2) \Rightarrow \lambda = b^y \]

linearize recursion in RG flow

\[ \delta K = K - K_c = J \frac{K}{k_B T} - J \frac{1}{k_B T_c} \]
\[ = \frac{J}{k_B T} \frac{T_c - T}{T_c} = k_B T \]

\[ \delta K' = \lambda \delta K = b^y \delta K ; \quad \delta K' = K' - K_c \text{ and } \delta K = K - K_c \]

because of composition law of RG

\[ \delta K = \lambda \delta K = b^y \delta K ; \quad \delta K' = K' - K_c \text{ and } \delta K = K - K_c \]

\[ f_s(K_c + \delta K) = b^{-d} f_s(K_c + b^y \delta K) \]
\[ f_s(\delta K) = b^{-d} f_s(b^y \delta K) \Rightarrow f_s(t) = b^{-d} f_s(b^y t) \]

but \( K_c \) is constant

\[ b = |t|^{1/y} \]

valid for any \( b \), so choose this parameterization

\[ \alpha = 0.131 \]

\[ f_s(t) = |t|^{d/y} f_s(t/|t|) \]
\[ f_s \sim |t|^{2-\alpha} ; \quad d/y = 2 - \alpha \]
\[ f_s(t) = |t|^{2-\alpha} f_s(t/|t|) \]

\[ y = \frac{\ln \lambda}{\ln b} = \frac{1}{\ln b} \ln \left( \frac{dR}{dK} \bigg|_{K=K_c} \right) = \frac{\ln \left( \frac{3}{2} \tanh 4K_c \right)}{\ln b} \]

\[ \delta K = K - K_c = J \frac{K}{k_B T} - J \frac{1}{k_B T_c} \]

\[ = \frac{J}{k_B T} \frac{T_c - T}{T_c} = k_B T \]

\[ \delta K' = \lambda \delta K = b^y \delta K ; \quad \delta K' = K' - K_c \text{ and } \delta K = K - K_c \]

\[ f_s(K_c + \delta K) = b^{-d} f_s(K_c + b^y \delta K) \]
\[ f_s(\delta K) = b^{-d} f_s(b^y \delta K) \Rightarrow f_s(t) = b^{-d} f_s(b^y t) \]

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\[ f_s(t) = |t|^{2-\alpha} f_s(t/|t|) \]

\[ y = \frac{\ln \lambda}{\ln b} = \frac{1}{\ln b} \ln \left( \frac{dR}{dK} \bigg|_{K=K_c} \right) = \frac{\ln \left( \frac{3}{2} \tanh 4K_c \right)}{\ln b} \]
Flow in the Case of More Than One Coupling Constant
Critical exponents associated with a fixed point are calculated by linearizing RG recursion relations about that fixed point:

\[ K_i' = R_i(K_1, K_2, \ldots, K_n) \]

\[ \delta K_i = \sum_{j=1}^{n} \frac{\partial R_i}{\partial K_j} \bigg|_{(K_i) = (K'_i)} \delta K_j = \sum_{j=1}^{n} M_{ij} \delta K_j \]

The form of the singular part of the free energy is a generalized homogeneous function, where exponents \( y_1, \ldots, y_n \) can be related to critical exponents.

The exponents \( y_1, \ldots, y_n \) determine the behavior of \( U_1, \ldots, U_n \) under the repeated action of the linear RG recursion relations. If \( y_i > 0 \), \( U_i \) is called relevant. If \( y_i < 0 \), \( U_i \) is called irrelevant and if \( y_i = 0 \), \( U_i \) is called marginal. We see that if a relevant scaling field is non-zero initially, then the linear recursion relations will transform this quantity away from the critical point. Alternatively, an irrelevant scaling field will transform this quantity toward the critical point, while marginal variable will be left invariant. Thus, relevant quantities must vanish at a critical point.

The existence of the relevant, irrelevant, and marginal variables explains the observation of universality, namely, that ostensibly disparate systems (e.g., fluids and magnets) show the same critical behavior near a second-order phase transition, including the same exponents for analogous physical quantities. In the RG picture, critical behavior is described entirely in terms of the relevant variables, while the microscopic differences between systems is due to irrelevant variables.
Do We Need Quantum Mechanics to Understand Phase Transitions at Finite Temperature?

Although quantum mechanics is essential to understand the existence of ordered phases of matter (e.g., superconductivity and magnetism are genuine quantum effects), it turns out that quantum mechanics does not influence asymptotic critical behavior of finite temperature phase transitions:

The decay time of temporal correlations for order-parameter fluctuations in dynamic (time-dependent) phenomena in the vicinity of critical point → critical slowing down

\[ \tau_c \sim \xi^z \sim |t|^{-\nu z} \]

In quantum systems static and dynamic fluctuations are not independent because the Hamiltonian determines not only the partition function, but also the time evolution of any observable via the Heisenberg equation of motion

\[ i\hbar \frac{d\hat{A}}{dt} = [\hat{A}, \hat{H}] \]

Thus, in quantum systems energy associated with the correlation time is also the typical fluctuation energy for static fluctuations, and it vanishes in the vicinity of a continuous phase transition as a power law

\[ E_c = \hbar / \tau_c \sim |t|^{\nu z} \]

This condition is always satisfied sufficiently close to \( T_c \), so that quantum effects are washed out by thermal excitations and a purely classical description of order parameter fluctuations is sufficient to calculate critical exponents

Phase transitions in classical models are driven only by thermal fluctuations, as classical systems usually freeze into a fluctuationless ground state at \( T = 0 \).
Formal Definition of Quantum Phase Transitions

- Quantum systems have fluctuations driven by the Heisenberg uncertainty in the ground state, and these can drive non-trivial phase transitions at $T = 0$.

$$\hat{H}(g) = \hat{H}_0 + g\hat{H}_1, \quad [\hat{H}_0, \hat{H}_1] = 0$$

- An avoided level-crossing between the ground and an excited state in a finite lattice could become progressively sharper as the lattice size increases, leading to a nonanalyticity at $g = g_c$ in the infinite lattice limit.

**DEFINITION:** Any point of nonanalyticity in the ground state energy of the infinite lattice system signifies quantum phase transition. The nonanalyticity could be either the limiting case of an avoided level-crossing or an actual level-crossing. QPT is usually accompanied by a qualitative change in the nature of the correlations in the ground state as one changes parameter in the Hamiltonian.
Simple Theoretical Model I: Quantum Criticality in Quantum Ising Chain (CoNb$_3$O$_3$)

Quantum Ising chain in the transverse external magnetic field

\[ \hat{H} = -J \sum_i \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z - g \sum_i \hat{\sigma}_i^x \]

\[ J > 0, \ g > 0 \]

Each ion has two possible states:

\[ \hat{\sigma}_i^z |\uparrow\rangle = + |\uparrow\rangle, \quad \hat{\sigma}_i^z |\downarrow\rangle = - |\downarrow\rangle \]

The first term in the Hamiltonian prefers that the spins on neighboring ions are parallel to each other, whereas the second allows quantum tunneling between the $|\uparrow\rangle$ and $|\downarrow\rangle$ states with amplitude proportional to $g$.
The two noncritical ground states of the dimer antiferromagnet have very different excitation spectra:

- Spin waves with nearly zero energy
- Oscillations of the magnitude of local magnetization

Mathematical expressions:

\[ |0\rangle = \prod_i |s_i\rangle \]

\[ |s_i\rangle = (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2} \]

\[ |t_{1i}\rangle = |\uparrow\uparrow\rangle \]

\[ |t_{0i}\rangle = (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)/\sqrt{2} \]

\[ |t_{-i}\rangle = |\downarrow\downarrow\rangle \]

\[ |t_m(k)\rangle = \frac{1}{\sqrt{N_d}} \sum_i e^{ik\cdot r_i} |t_m\rangle_i \prod_{j\neq i} |s_j\rangle \]

How Quantum Criticality Extends to Non-Zero Temperatures

For small $g$, thermal effects induce spin waves that distort the Néel antiferromagnetic ordering. For large $g$, thermal fluctuations break dimers in the blue region and form quasiparticles called triplons. The dynamics of both types of excitations can be described quasi-classically.

Quantum criticality appears in the intermediate orange region, where there is no description of the dynamics in terms of either classical particles or waves. Instead, the system exhibits the strongly coupled dynamics of nontrivial entangled quantum excitations of the quantum critical point $g_c$.

Wavefunction at $g=g_c$ is a complex superposition of an exponentially large set of configurations fluctuating at all length scales → thus, it cannot be written down explicitly due to long-range quantum entanglement which emerges for a very large number of electrons and between electrons separated at all length scales.
Scaling in the Vicinity of Quantum Critical Points

Physics is dominated by thermal excitations of the quantum critical ground state.

\[ k_B T = \Delta \propto |r - r_c|^{\nu z} \]

\[ Z = \int \mathcal{D}[\Phi] \exp \left\{- \int_0^{1/T} d\tau \int d^D r \mathcal{L}[\Phi(r, \tau)] \right\} \]

\[ \Phi(r, \tau) \]

represents fluctuations of the order parameter and it depends on the imaginary time \( \tau \) which takes values in the interval \([0, 1/T]\); the imaginary time direction acts like an extra dimension, which becomes infinite for \( T \to 0 \).

\[ T = 0 : \quad f_{\text{sing}}(G, h) = b^{-(D+z)} f_{\text{sing}}(b^{y_g} G, b^{y_h}) \]

\[ T > 0 : \quad f_{\text{sing}}(G, h, T) = b^{-(D+z)} f_{\text{sing}}(b^{y_g} G, b^{y_h}, b^z T) \]

\[ G = |g - g_c|/g_c \]

\[ 1/y_g = \nu \]
Example of Scaling Analysis: Superconductor-Insulator QPT in Thin Films

The success of finite-size scaling analyses of the superconductor-insulator transitions as a function of film thickness or applied magnetic field provides strong evidence that $T = 0$ quantum phase transitions are occurring.

$$R_c = \frac{\hbar}{4e^2} = 6450 \, \Omega$$


$$\xi \sim g^{-\nu} \quad \xi_T \sim \xi^z$$

$$R_\square = R_c f \left( \frac{g}{T^{1/z\nu}} \right)$$

$g = B$ or $g = d$

$z\nu \approx 1.4$

$z\nu \approx 1.36$
RG Coarse Graining for 1D Quantum Ising Model in Transverse Magnetic Field

\[ \hat{H} = \sum_{j \in \text{odd}} \hat{H}_{\text{block},j} + \sum_{j \in \text{even}} \hat{H}_{\text{coupling},j} \]

\[ \hat{H}_{\text{block},j} = \begin{pmatrix} -J & 0 & -h & 0 \\ 0 & J & 0 & -h \\ -h & 0 & J & 0 \\ 0 & -h & 0 & -J \end{pmatrix} \begin{pmatrix} |1\rangle \\ |2\rangle \\ |3\rangle \\ |4\rangle \end{pmatrix} = \begin{pmatrix} -\sqrt{J^2 + h^2} |1\rangle \\ -\sqrt{J^2 + h^2} |2\rangle \\ \sqrt{J^2 + h^2} |3\rangle \\ \sqrt{J^2 + h^2} |4\rangle \end{pmatrix} \]

\[ \hat{H}_{\text{coupling},j} = -J \hat{\sigma}_j^z \hat{\sigma}_j^z - h \hat{\sigma}_j^x \]

keep \( |1\rangle \) and \( |2\rangle \) only

\[ \hat{P}_I = |1\rangle \langle 1\rangle + |2\rangle \langle 2\rangle \]

\[ \tilde{\sigma}_z = |1\rangle \langle 1\rangle - |2\rangle \langle 2\rangle \]

\[ \tilde{\sigma}_x = |1\rangle \langle 2\rangle + |2\rangle \langle 1\rangle \]

\[ \hat{P} = \hat{P}_1 \otimes \hat{P}_2 \otimes \cdots \hat{P}_{N/2} \quad \text{projector onto coarse-grained system} \]

\[ \hat{P}_I \hat{H}_{\text{block},j} \hat{P}_I = -\sqrt{J^2 + h^2} \hat{P}_I \]

\[ \left( \hat{P}_I \otimes \hat{P}_{I+1} \right) \hat{H}_{\text{couple},j} \left( \hat{P}_I \otimes \hat{P}_{I+1} \right) = -J \left( \hat{P}_I \hat{\sigma}_j^z \hat{P}_I \right) \otimes \left( \hat{P}_{I+1} \hat{\sigma}_j^{j+1} \hat{P}_{I+1} \right) - h \left( \hat{P}_I \hat{\sigma}_j^x \hat{P}_I \right) \otimes \hat{P}_{I+1} \]
RG Flow for 1D Quantum Ising Model in Transverse Magnetic Field

\[ \hat{H} = -J \sum_{j=1}^{N} \hat{\sigma}_j^z \hat{\sigma}_{j+1}^z - h \sum_{j=1}^{N} \hat{\sigma}_j^x \]

\[ \tilde{H} = \hat{P} \hat{H} \hat{P} = -\sqrt{J^2 + h^2} \sum_{l=1}^{N/2} \tilde{P}_l - \frac{J^2}{\sqrt{J^2 + h^2}} \sum_{l=1}^{N/2-1} \tilde{\sigma}_l^z \tilde{\sigma}_{l+1}^z - \frac{h^2}{\sqrt{J^2 + h^2}} \sum_{l=1}^{N/2} \tilde{\sigma}_l^x \]

RG flow equations

\[ \begin{pmatrix} J' \n \h' \end{pmatrix} = \begin{pmatrix} \frac{J^2}{\sqrt{J^2 + h^2}} \\ \frac{h^2}{J} \sqrt{J^2 + h^2} \end{pmatrix} \quad \Leftrightarrow \quad K' = \frac{h'}{J'} = \left( \frac{h}{J} \right)^2 = K^2 \]

original Hamiltonian for N spins

course grained Hamiltonian for N/2 spins after one RG iteration

Linearize RG equations close to \( K_c \)

\[ K' = K_c \approx \left. \frac{dK'}{dK} \right|_c (K - K_c) \]

\[ \lambda_K = \left. \frac{dK'}{dK} \right|_c = 2 = b^{y_K} \Rightarrow y_K = 1 \]

Extract critical exponents from RG flow

\[ \xi(K') = \frac{1}{b} \xi(K) \Rightarrow \xi = \frac{\text{const}}{K - K_c} \propto (K - K_c)^{-\nu} \Rightarrow \nu = \frac{1}{y_K} = 1 \]

\[ \langle \tilde{\sigma}_z^1 \rangle = \frac{J}{\sqrt{J^2 + h^2}} \langle \tilde{\sigma}_z^1 \rangle = \frac{1}{\sqrt{1 + K^2}} \langle \tilde{\sigma}_z^1 \rangle \]

\[ m(K) = b^{-x} m(b^{y_K} K) \Rightarrow b^{-x} = \frac{1}{\sqrt{1 + K_c^2}} \Rightarrow x = \frac{1}{2}, \beta = 1 - x = \frac{1}{2} \]