Tutorial on Renormalization Group Applied to Classical and Quantum Critical Phenomena

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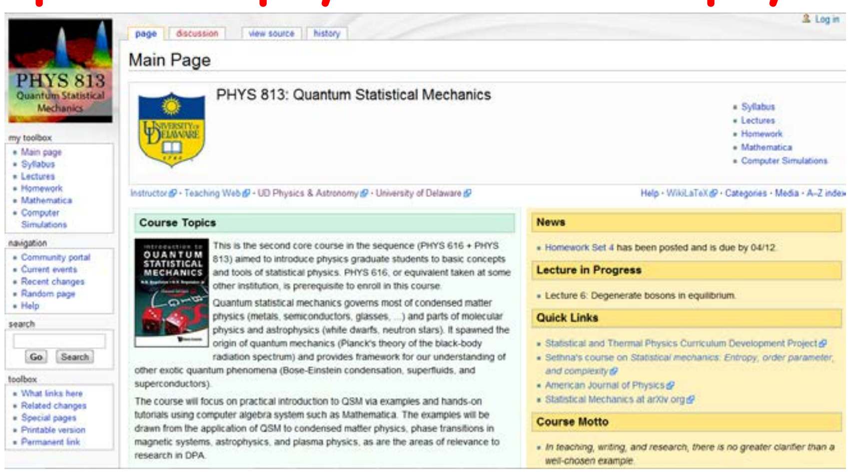
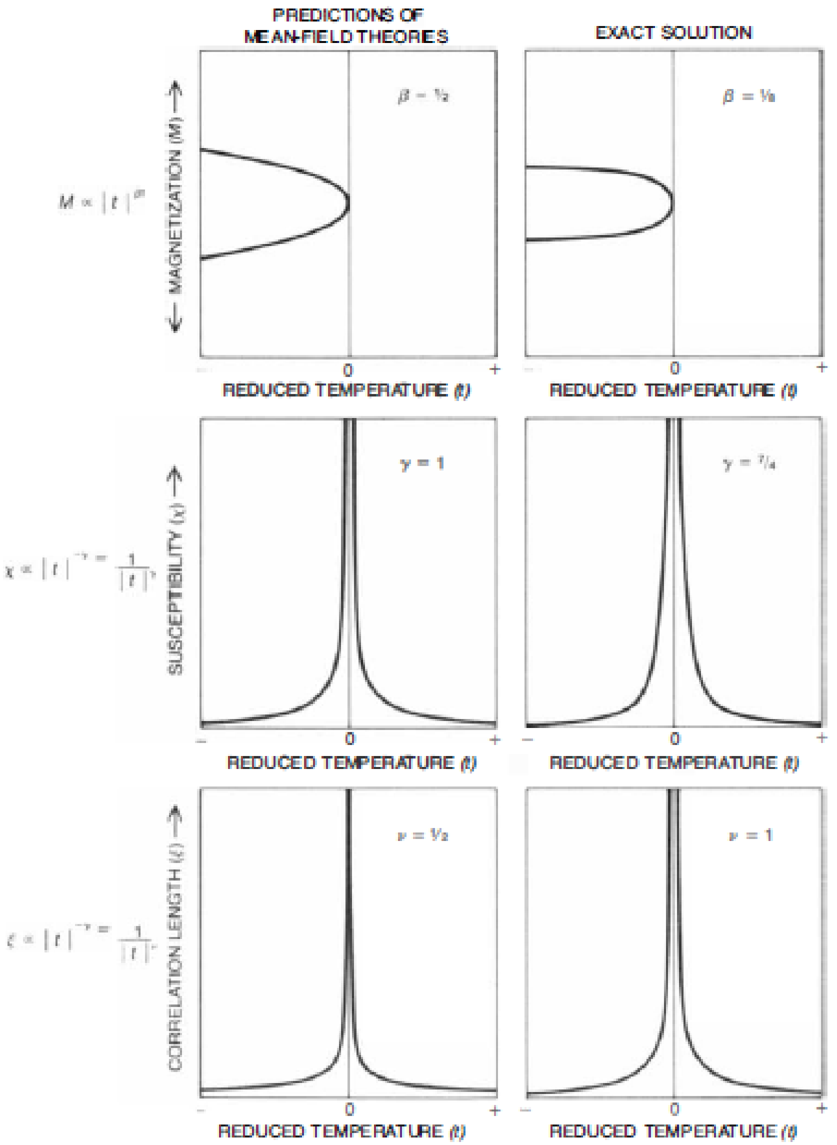
http://wiki.physics.udel.edu/phys813

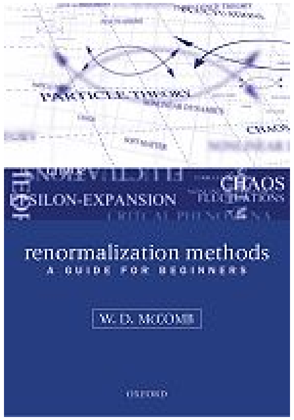


The Task for Renormalization Group (RG) Theory



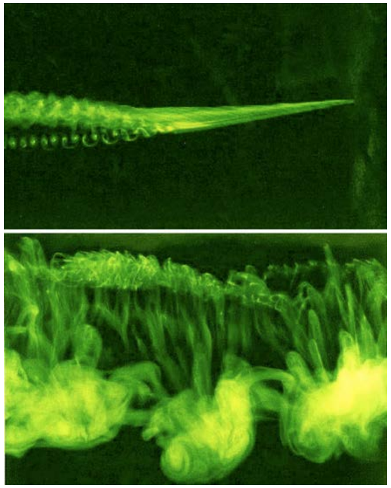
K. G. Wilson
"Problems in physics with many scales of length,"
Scientific American **241(2)**, 140 (1979)

RG was developed to understand:
- why divergences occur in quantum field theory (i.e., why parameters like the electron mass are scale dependent).
- universality and scaling in second-order or continuous phase transitions



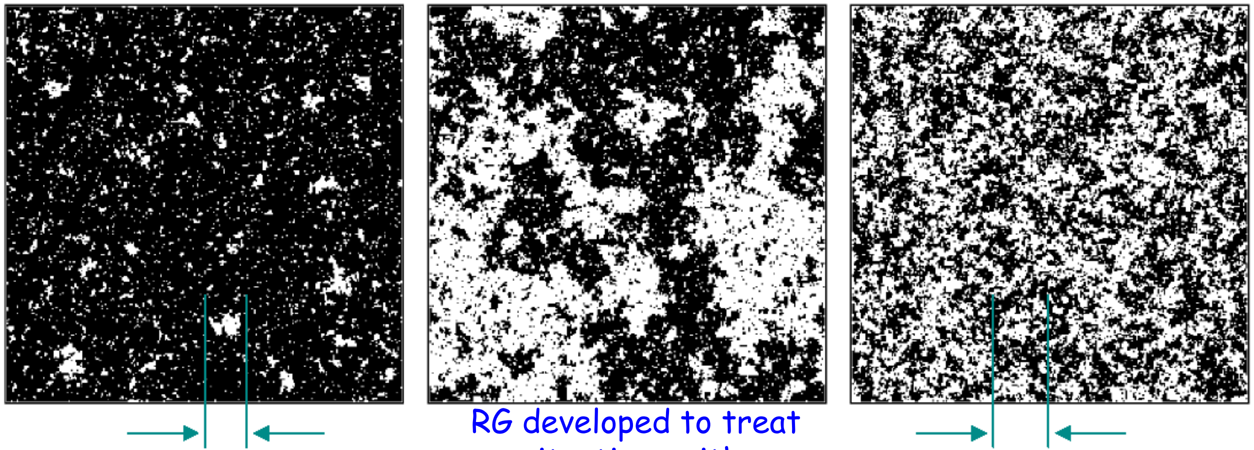
Other problems in physics with many scales of length can be handled by RG

Turbulence



Quick and Dirty Introduction Using Numerical Renormalization Group (RG) for Ising Model

$T < T_c$ $\qquad$ $T \simeq T_c$ $\qquad$ $T > T_c$

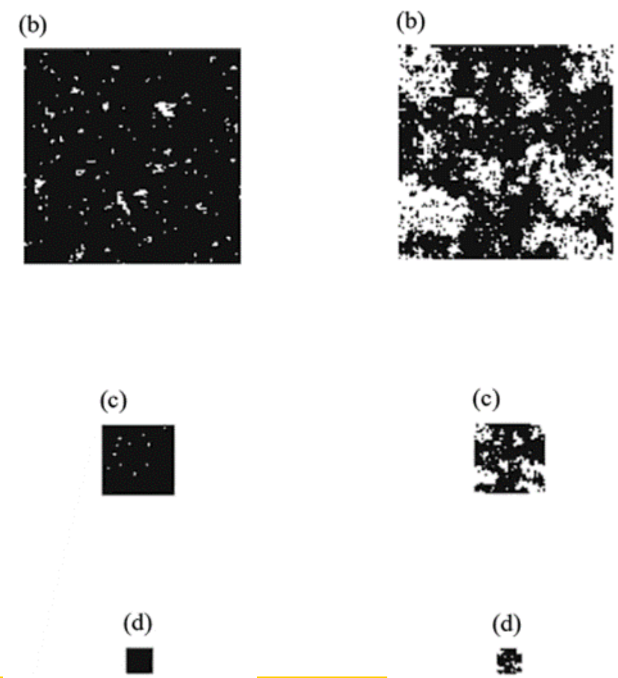


ξ

RG developed to treat situations with: $\xi \simeq \infty$

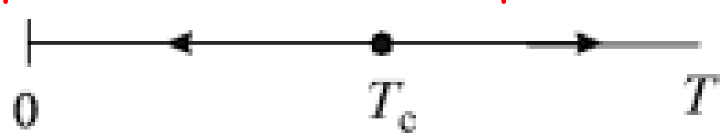
ξ

Result of RG procedure: A renormalized system which has the same long-wavelength properties as the original system, but fewer degrees of freedom.



Block spin transformation leads to RG flow

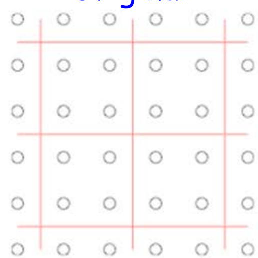
The RG flow changes the effective temperature, where the critical point corresponds to the unstable point of the flow

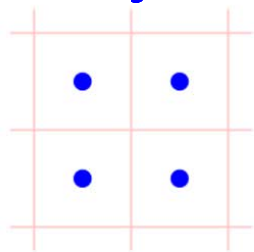


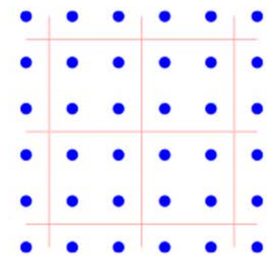
RG Steps in Pictures

Original | Coarse-grained | Rescaled

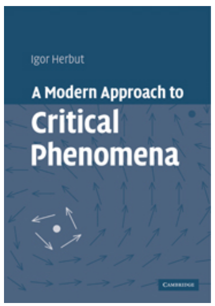
Real-Space

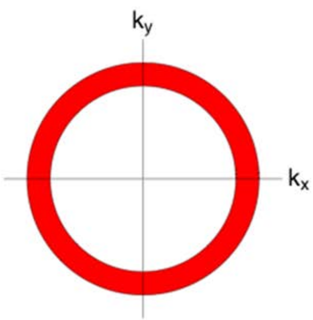


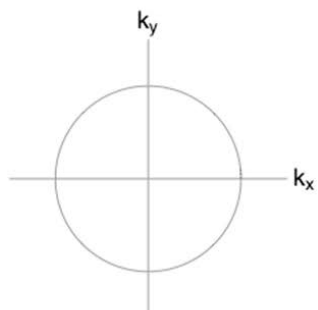


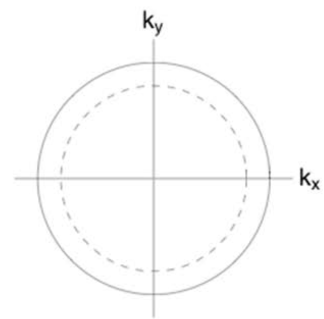


Momentum-Space









CONVENTIONS

$$H = -\frac{J}{k_B T}\sum_{\langle ij\rangle} S_i S_j = -K \sum_{\langle ij\rangle} S_i S_j$$

$$Z = \sum_{\{S_i\}} e^{-H(\{S_i\})}$$

VOCABULARY

$$Z_N(H) = \sum_{\{S_i\}} \sum_{\{S_i'\}} e^{-H(\{S_i, S_i'\})} = \sum_{\{S_i'\}} e^{-H'(\{S_i'\})} = Z_{N'}(H')$$ coarse-graining via partial evaluation of partition function

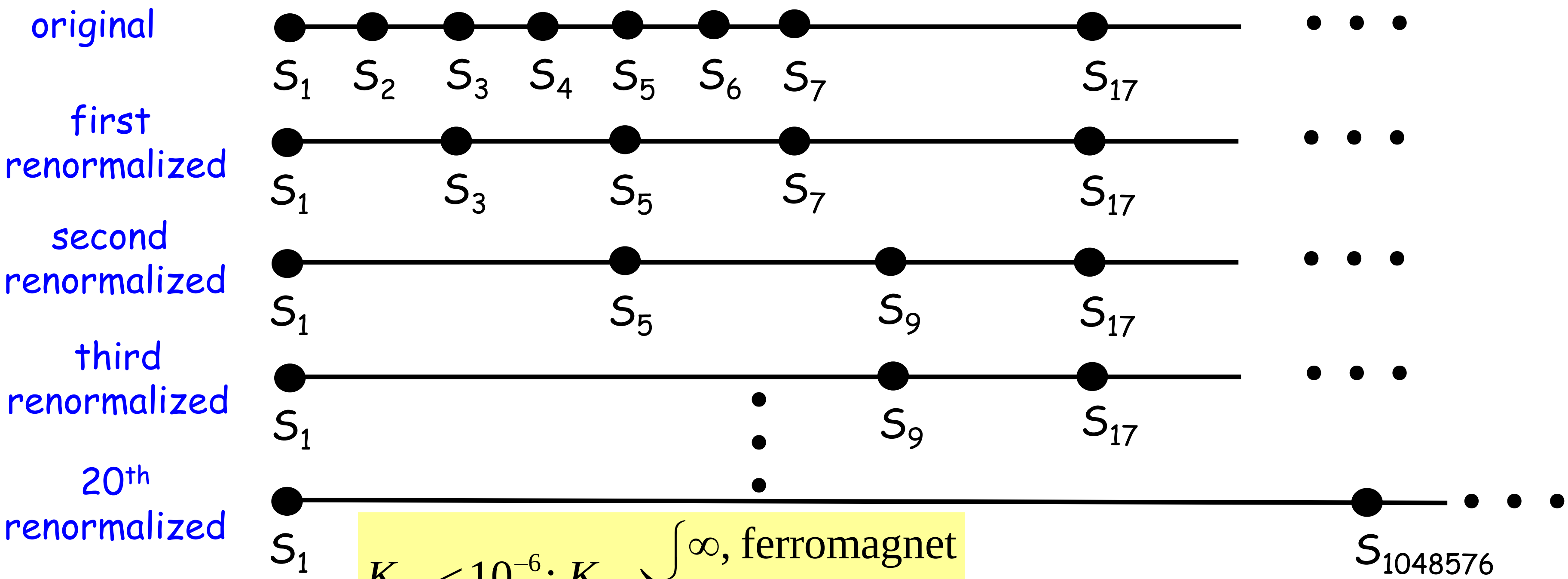
$$H' = R_b(H)$$ RG recursion relations

$$H'' = R_{b_2}(H') = R_{b_2} \cdot R_{b_1}(H) = R_{b_1 b_2}(H)$$ RG is semi-group

$$H^* = R_b(H^*),\ H^* = \lim_{n\to\infty} R_b^n(H_c)$$ fixed point definition

$$r' = b^{-1} r,\ N' = b^{-d} N,\ f(H') = b^d f(H)$$ rescaling

Real-Space RG for 1D Ising Model



$$K_{20} < 10^{-6};\ K \to \begin{cases} \infty, \text{ ferromagnet} \\ 0, \text{ paramagnet} \end{cases}$$

Teaching the renormalization group[a]

Humphrey J. Maris and Leo P. Kadanoff
Department of Physics, Brown University, Providence, Rhode Island 02912
(Received 24 August 1977; accepted 29 October 1977)

The renormalization group theory of second-order phase transitions is described in a form suitable for presentation as part of an undergraduate statistical physics course.

K	$\zeta(K)$ Renormalization group	Exact
0.01	ln 2	0.693 197
0.100 334	0.698 147	0.698 172
0.327 447	0.745 814	0.745 827
0.636 247	0.883 204	0.883 210
0.972 710	1.106 299	1.106 302
1.316 710	1.386 078	1.386 080
1.662 637	1.697 968	1.697 968
2.009 049	2.026 876	2.026 877
2.355 582	2.364 536	2.364 537
2.702 146	2.706 633	2.706 634

RG Flow for 1D Ising Model

$$Z_N(K)=\sum_{\{S_i\}} e^{K(S_1S_2+S_2S_3+\dots)}=\sum_{\{S_i\}}\left[e^{KS_1S_2}e^{KS_2S_3}\right]\left[e^{KS_3S_4}e^{KS_4S_5}\right]\cdots$$

$$=\sum_{\{S_i\neq S_2\}}\left[e^{KS_1+KS_3}+e^{-KS_1-KS_3}\right]\left[e^{KS_3S_4}e^{KS_4S_5}\right]\cdots$$

$$e^{KS_1+KS_3}+e^{-KS_1-KS_3}=A(K)e^{B(K)S_1S_3}$$

$$\underbrace{\begin{aligned}&S_1=S_3=\pm1\Rightarrow 2\cosh\left(2K\right)=A(K)e^{B(K)}\\&S_1=-S_3=\pm1\Rightarrow 2=A(K)e^{-B(K)}\end{aligned}}_{\Downarrow}$$

$$A(K)=2\sqrt{\cosh\left(2K\right)};\ B(K)=\frac{1}{2}\ln\cosh\left(2K\right)=K_1$$

$$Z_N(K)=\left[A(K)\right]^{N/2}Z_{N/2}(K_1)$$

$$\underbrace{K_{i+1}=\frac{1}{2}\ln\cosh\left(2K_i\right)}_{\Downarrow}$$

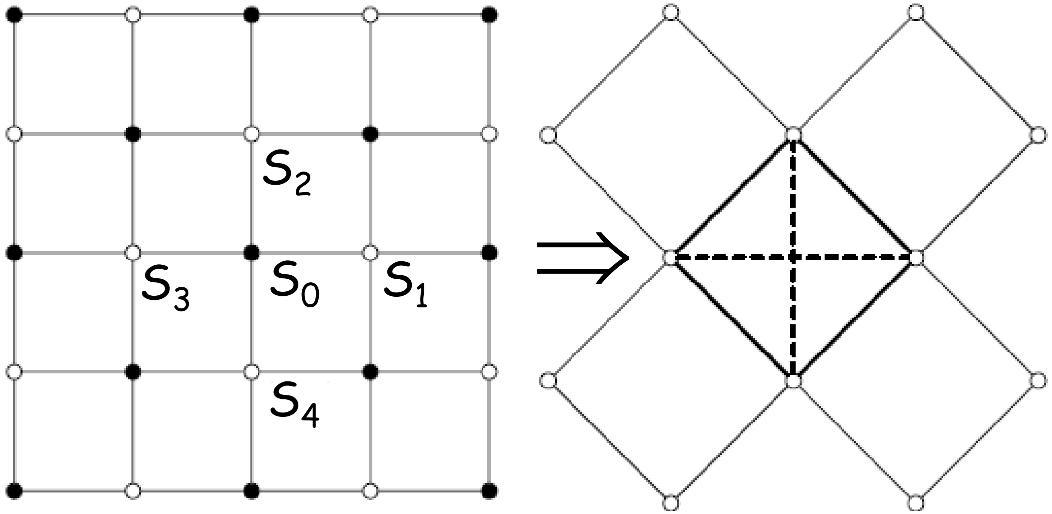
$$K^*=\frac{1}{2}\ln\cosh\left(2K^*\right)$$

stable fixed point 0

RG flow does not contain non-trivial fixed points, which means no phase transition at any finite temperature

∞ unstable fixed point

Real-Space RG for 2D Ising Model



$$Z_N(K) = \sum_{\{S_i\}} e^{K(S_0S_1 + S_0S_2 + S_0S_3 + S_0S_4 \ldots)}$$
$$= \sum_{\{S_i \neq S_0\}} \left[e^{K(S_1+S_2+S_3+S_4)} + e^{-K(S_1+S_2+S_3+S_4)} \right] \cdots$$

$$e^{K(S_1+S_2+S_3+S_4)} + e^{-K(S_1+S_2+S_3+S_4)} = A(K) e^{B(K)[S_1S_2+S_2S_3+S_3S_4+S_4S_1+S_1S_3+S_2S_4]+C(K)S_1S_2S_3S_4}$$

$$\underbrace{\begin{aligned} &S_1 = S_2 = S_3 = S_4 = \pm 1 \Rightarrow 2\cosh 4K = A(K)e^{6B(K)+C(K)} \\ &S_1 = S_2 = S_3 = -S_4 = \pm 1 \Rightarrow 2\cosh 2K = A(K)e^{-C(K)} \\ &S_1 = S_2 = -S_3 = -S_4 = \pm 1 \Rightarrow 2 = A(K)e^{-2B(K)+C(K)} \end{aligned}}_{\Downarrow}$$

$$A(K) = 2\sqrt{\cosh 2K}\sqrt[8]{\cosh 4K};\ B(K) = \frac{1}{8}\ln\cosh 4K;\ C(K) = \frac{1}{8}\ln\cosh 4K - \frac{1}{2}\ln\cosh 2K$$

$$Z_N(K) = [A(K)]^{N/2} Z_{N/2}(K_1, K_2, K_3)$$

$$H = -2B(K)\sum_{\langle ij \rangle} S_iS_j - B(K)\sum_{[ij]} S_iS_j - C(K)\sum_{square} S_iS_jS_kS_l$$

If we set K_2=K_3=0, we get similar trivial RG flow as in 1D Ising model:

$$K_1^{i+1} = \frac{1}{4}\ln\cosh 4K_1^i$$

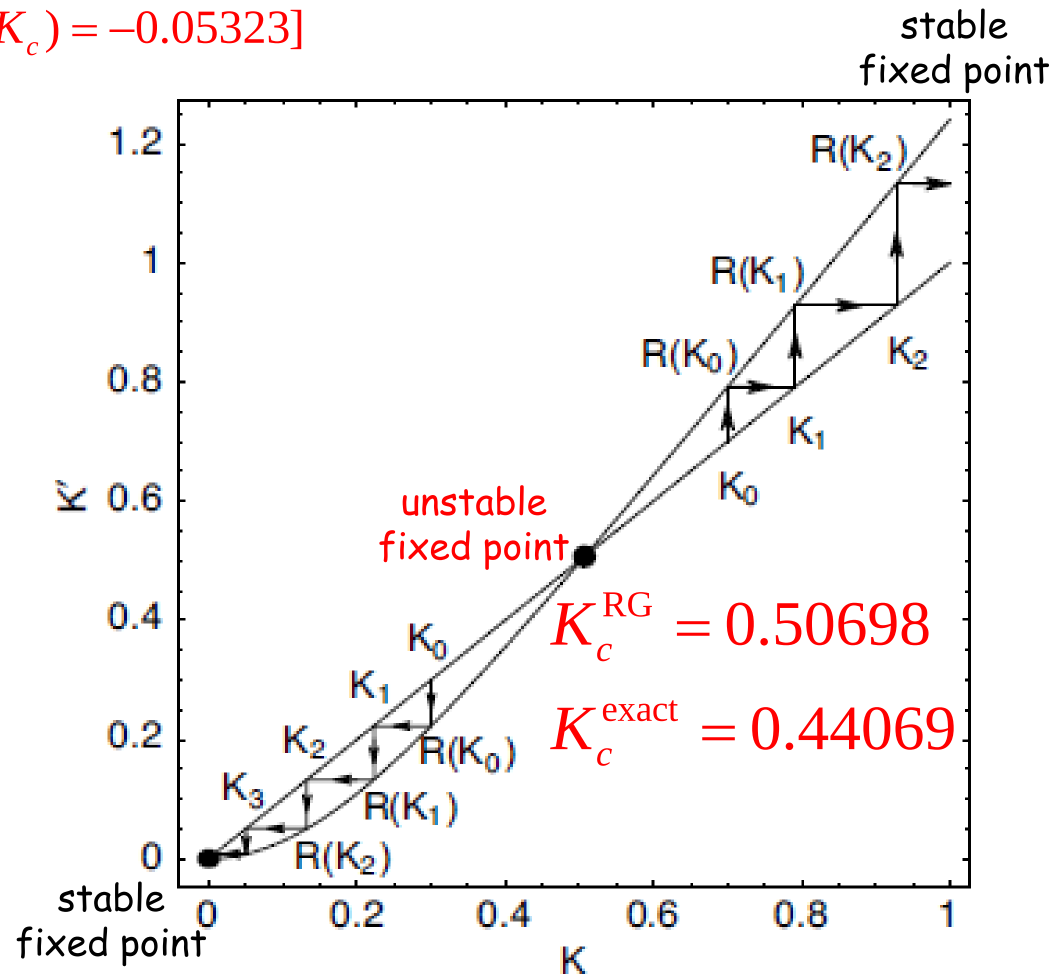
RG Flow for 2D Ising Model

0 [a posteriori we indeed find $K_3'(K_c) = -0.05323$]

$$K_1' = \tfrac{1}{4}\ln(\cosh 4K)\,,$$
$$K_2' = \tfrac{1}{8}\ln(\cosh 4K)\,,$$
$$\cancel{K_3' = \tfrac{1}{8}\ln(\cosh 4K) - \tfrac{1}{2}\ln(\cosh 2K)}$$

Both K_1 and K_2 are positive and thus favor the alignment of spins. Hence, we will omit the explicit presence of K_2 in the renormalized energy, but increase K_1 accordingly to a new value $K' = K'_1 + K'_2$ so that the tendency toward alignment is approximately the same

$$\boxed{\begin{aligned} K' &= K_1' + K_2' \\ K' &= \tfrac{3}{8}\ln(\cosh 4K) \end{aligned}}$$



$$K_c^{\mathrm{RG}} = 0.50698$$
$$K_c^{\mathrm{exact}} = 0.44069$$

Scaling Hypothesis for Free Energy

- Thermodynamic exponents α, β, γ, δ are not independent and can be obtained from exponents ν and η governing the scaling of the correlation function (Widom 1965 and Kadanoff 1966 prior to RG)

$$f(t,h) = f_{\mathrm{sing}}(t,h) + f_{\mathrm{reg}}(t,h)$$

close to critical point free energy can be decomposed in singular and regular part

$$f_{\mathrm{sing}}(t,h) = b^{-d} f_{\mathrm{sing}}(b^{y_t} t, b^{y_h} h)$$

scaling hypothesis assumes that singular part is generalized homogeneous function

$$b^{y_t} = 1/|t| \Rightarrow f_{\mathrm{sing}}(t,h) = |t|^{d/y_t} \Phi_{\pm}\left(\frac{h}{|t|^{y_h/y_t}}\right)$$

$$\Phi_{\pm}(x) = f_{\mathrm{sing}}(\pm 1, x)$$

b is an arbitrary (dimensionless) scale factor and exponents y_t and y_h are characteristic of universality class

thermodynamic **scaling relations** which can be tested experimentally

$$2-\alpha = 2\beta + \gamma$$

$$2-\alpha = \beta(\delta+1)$$

$$C \approx \frac{1}{T_c}\frac{\partial^2 f_{\mathrm{sing}}}{\partial t^2} = |t|^{\frac{D}{y_t}-2} = |t|^{-\alpha} \Rightarrow \alpha = 2 - \frac{d}{y_t}$$

$$m \approx -\left.\frac{\partial f_{\mathrm{sing}}}{\partial h}\right|_{h=0} \propto (-t)^{\frac{D-y_h}{y_t}} = (-t)^{\beta} \Rightarrow \beta = \frac{d-y_h}{y_t}$$

$$m(t,h) \approx -\frac{\partial f_{\mathrm{sing}}}{\partial h} = |t|^{\frac{D-y_h}{y_t}} \Phi'_{\pm}\left(\frac{h}{|t|^{\frac{y_h}{y_t}}}\right) \overset{\substack{t\to 0\\ x\to\infty\\ m<\infty}}{\propto} |t|^{\frac{d-y_h}{y_t}} x^{\frac{d}{y_h}-1} \Rightarrow m(h) \propto h^{\frac{D}{y_h}-1} = h^{1/\delta}, \delta = \frac{y_h}{d-y_h}$$

$$\chi \approx \frac{\partial^2 f_{\mathrm{sing}}}{\partial h^2} \propto (-t)^{\frac{D-2y_h}{y_t}} = (-t)^{-\gamma} \Rightarrow \gamma = \frac{2y_h - d}{y_t}$$

Scaling Hypothesis for Correlation Function

$$G_{\text{sing}}(|\mathbf{r}|;t,h) = b^{-2(d-y_h)} G_{\text{sing}}\left(\frac{|\mathbf{r}|}{b}; b^{y_t} t, b^{y_h} h\right)$$

close to the critical point correlation function is dominated by its singular part → actually valid only if the dimension of space of smaller than certain upper critical dimension otherwise fails due to the so-called **dangerously irrelevant couplings in RG**

$$b = |t|^{-1/y_h} \Rightarrow G_{\text{sing}}(|\mathbf{r}|;t,0) = |t|^{\frac{2(d-y_h)}{y_t}} \Psi_{\pm}\left(\frac{|\mathbf{r}|}{|t|^{-1/y_t}}\right)$$

$$\Psi_{\pm}(x) = G_{\text{sing}}(x;\pm 1,0) \overset{\substack{|t|\neq 0 \\ |\mathbf{r}|\to\infty}}{\Rightarrow} G_{\text{sing}}(|\mathbf{r}|) \propto e^{-|\mathbf{r}|/\xi} \Rightarrow \xi \propto |t|^{-1/y_t} = |t|^{-\nu}$$

$$\nu = \frac{1}{y_t} \Rightarrow 2 - \alpha = d\nu$$

hyperscaling relation because it connects singularities in thermodynamic observables with singularities related to the correlation function

$$|t| \to 0,\ G_{\text{sing}}(|\mathbf{r}|;t,0) < \infty \text{ iff } \Psi_{\pm}(x) \overset{x\to\infty}{\propto} |x|^{-2(d-y_h)}$$

$$G_{\text{sing}}(|\mathbf{r}|;0,0) \propto |\mathbf{r}|^{-2(d-y_h)} = |\mathbf{r}|^{-(d-2+\eta)}$$

$$2(d-y_h) = d-2+\eta \Leftrightarrow \eta = d+2-2y_h$$

$\gamma = \nu(2-\eta)$ another **hyperscaling** relation

four thermodynamic exponents expressed in terms of the two correlation function exponents

$$\alpha = 2 - d\nu$$

$$\beta = \frac{\nu}{2}(d-2+\eta)$$

$$\gamma = \nu(2-\eta)$$

$$\delta = \frac{d+2-\eta}{d-2+\eta}$$

Scaling Hypothesis Applied to Extract Critical Exponents via RG for 2D Ising Model

$$\boxed{C_V = -T\left(\frac{\partial^2 F}{\partial T^2}\right)_V \qquad C_V \sim |T - T_c|^{-\alpha} \sim |t|^{-\alpha}}$$

$$\ln \mathcal{Z} = N f(K)$$

$$f(K') = 2f(K) - \ln\left[2(\cosh 4K)^{1/8}(\cosh 2K)^{1/2}\right]$$

removal of short-range degrees of freedom results in an expression that is analytic in K

$$\delta K = K - K_c = \frac{J}{k_B T} - \frac{J}{k_B T_c} = \frac{J}{k_B T}\frac{T_c - T}{T_c} = Kt$$

$$f_s(K) = b^{-d} f_s(K'); \qquad b = \sqrt{2} \text{ and } d = 2$$

focus on the singular part of free energy

$$K' = R(K) = K_c + \left.\frac{dR}{dK}\right|_{K=K_c}(K - K_c) + \cdots$$

linearize recursion in RG flow

$$\lambda = R'|_{K_c};\ \lambda(b)\lambda(b) = \lambda(b^2) \Rightarrow \lambda = b^y$$

can be written like this because of composition law of RG

$$\delta K' = \lambda\,\delta K = b^y\,\delta K\,; \quad \delta K' = K' - K_c \text{ and } \delta K = K - K_c$$

$$f_s(K_c + \delta K) = b^{-d} f_s(K_c + b^y\,\delta K)$$

but K_c is constant

$$f_s(\delta K) = b^{-d} f_s(b^y\,\delta K) \Rightarrow f_s(t) = b^{-d} f_s(b^y t)$$

K remains finite at the critical point

$$b = |t|^{1/y}$$

valid for any b, so choose this parameterization

$$f_s(t) = |t|^{d/y} f_s(t/|t|)$$

$$f_s \sim |t|^{2-\alpha},\ d/y = 2 - \alpha$$

$$f_s(t) = |t|^{2-\alpha} f_s(t/|t|)$$

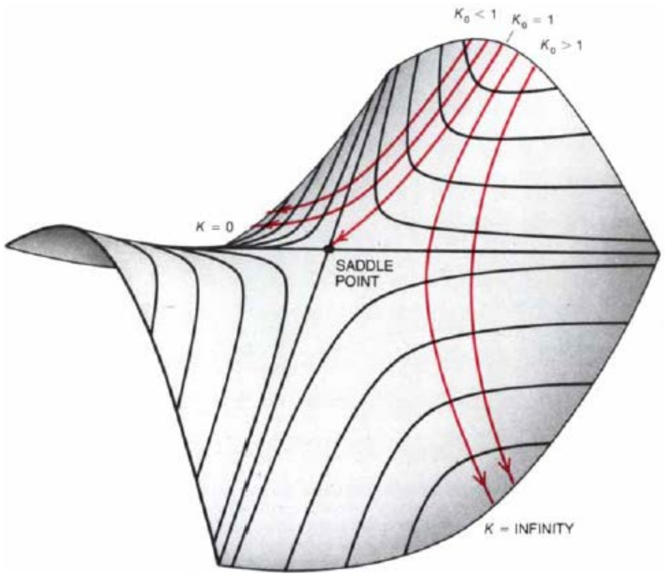
$$y = \frac{\ln\lambda}{\ln b} = \frac{1}{\ln b}\ln\left(\left.\frac{dR}{dK}\right|_{K=K_c}\right) = \frac{\ln(\frac{3}{2}\tanh 4K_c)}{\ln b}$$

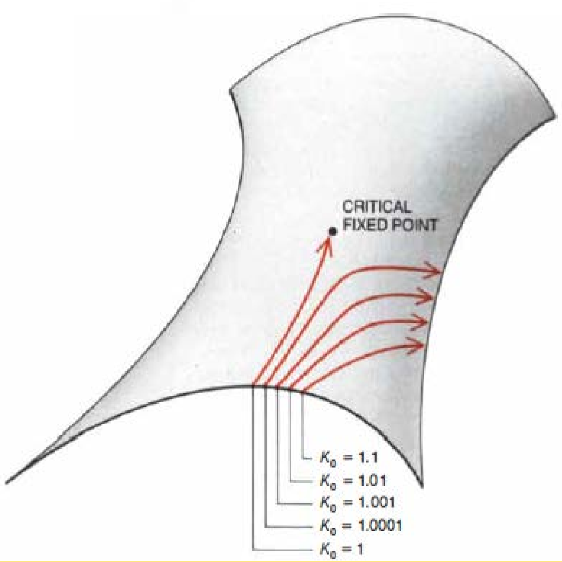
α=0.131

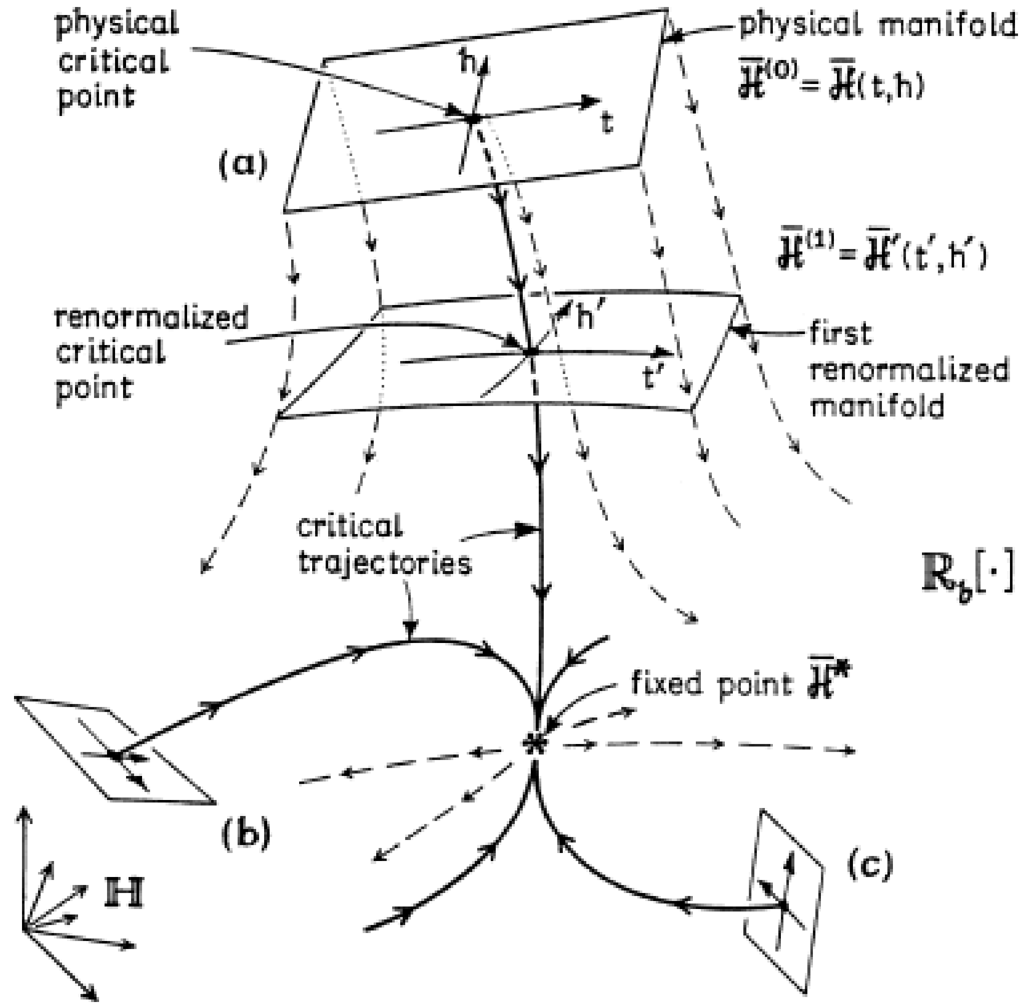
$$m(t, h=0) = \left.\frac{\partial f(t,h)}{\partial h}\right|_{h=0} \propto t^{(d-y_h)/y}$$

$$\chi(t, h=0) = \left.\frac{\partial f^2(t,h)}{\partial h^2}\right|_{h=0} \propto t^{(d-2y_h)/2y}$$

Flow in the Case of More Than One Coupling Constant







Relevant, Irrelevant and Marginal Variables in RG Flow

❑ Critical exponents associated with a fixed point are calculated by linearizing RG recursion relations about that fixed point:

$$K_i' = R_i(K_1, K_2, \ldots, K_n)$$

$$\delta K_i' = \sum_{j=1}^{n} \left.\frac{\partial R_i}{\partial K_j}\right|_{\{K_i\}=\{K_i^*\}} \delta K_j \equiv \sum_{j=1}^{n} M_{ij}\delta K_j$$

The eigenvectors and eigenvectors of the matrix **M** with entries M_{ij}, $\mathbf{M}U_i = \lambda_i U_i = b^{y_i} U_i$, yield the linear **scaling fields** U_i, for $i = 1, 2, \ldots, n$ in terms of which the singular part of the free energy is expressed as

$$f_s(U_1, U_2, \ldots, U_n) = b^{-d} f_s(b^{y_1}U_1, b^{y_2}U_2, \ldots, b^{y_n}U_n)\,. \tag{8.25}$$

❑ The form of the singular part of the free energy is a **generalized homogeneous function**, where exponents $y_1, \ldots, y_n$ can be related to critical exponents

❑ The exponents $y_1, \ldots, y_n$ determine the behavior of $U_1, \ldots, U_n$ under the repeated action of the linear RG recursion relations. If $y_i > 0$, U_i is called **relevant**. If $y_i < 0$, U_i is called **irrelevant** and if $y_i = 0$, U_i is called **marginal**. We see that if a relevant scaling field is non-zero initially, then the linear recursion relations will transform this quantity away from the critical point. Alternatively, an irrelevant scaling field will transform this quantity toward the critical point, while marginal variable will be left invariant. Marginal variable is associated with logarithmic corrections to scaling.

❑ The existence of the relevant, irrelevant, and marginal variables explains the observation of universality, namely, that ostensibly disparate systems (e.g., fluids and magnets) show the same critical behavior near a second-order phase transition, including the same exponents for analogous physical quantities. In the RG picture, critical behavior is described entirely in terms of the relevant variables, while the microscopic differences between systems is due to irrelevant variables.

Do We Need Quantum Mechanics to Understand Phase Transitions at Finite Temperature?

- Although quantum mechanics is essential to understand the existence of ordered phases of matter (e.g., superconductivity and magnetism are genuine quantum effects), it turns out that quantum mechanics does not influence asymptotic critical behavior of finite temperature phase transitions:

$$\tau_c \sim \xi^z \sim |t|^{-\nu z}$$

The decay time of temporal correlations for order-parameter fluctuations in dynamic (time-dependent) phenomena in the vicinity of critical point $\longrightarrow$ **critical slowing down**

$$i\hbar \frac{d\hat{A}}{dt} = [\hat{A}, \hat{H}]$$

In quantum systems static and dynamic fluctuations are not independent because the Hamiltonian determines not only the partition function, but also the time evolution of any observable via the Heisenberg equation of motion

$$E_c = \hbar/\tau_c \sim |t|^{\nu z}$$

Thus, in quantum systems energy associated with the correlation time is also the typical fluctuation energy for static fluctuations, and it vanishes in the vicinity of a continuous phase transition as a power law

$$E_c \ll k_B T_c$$

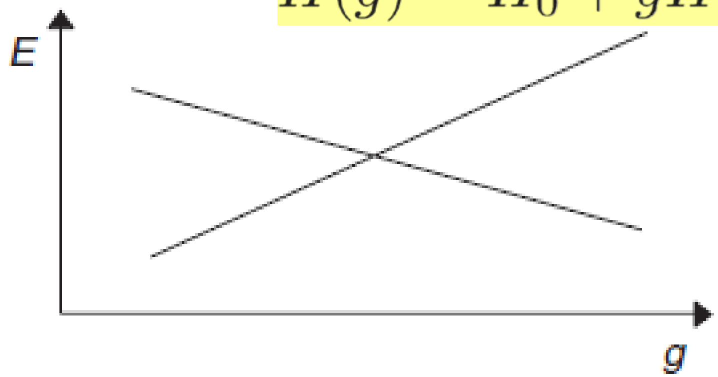
This condition is always satisfied sufficiently close to T_c, so that quantum effects are washed out by thermal excitations and a purely classical description of order parameter fluctuations is sufficient to calculate critical exponents

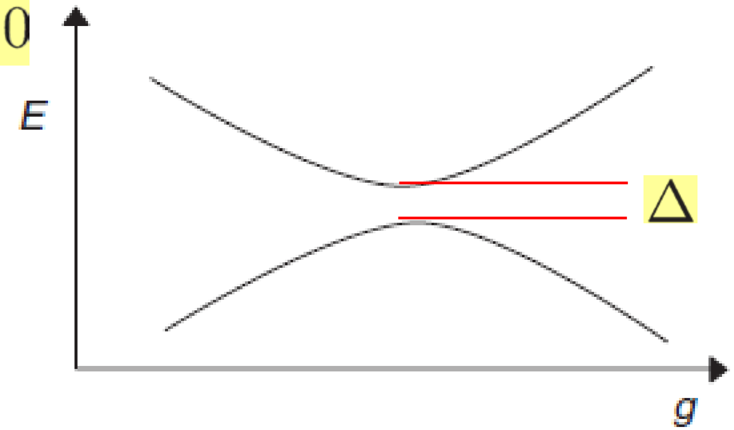
- Phase transitions in classical models are driven only by thermal fluctuations, as classical systems usually freeze into a fluctuationless ground state at $T = 0$

Formal Definition of Quantum Phase Transitions

❑Quantum systems have fluctuations driven by the Heisenberg uncertainty in the ground state, and these can drive non-trivial phase transitions at $T = 0$

$$\hat{H}(g) = \hat{H}_0 + g\hat{H}_1, \;\; [\hat{H}_0, \hat{H}_1] = 0$$





❑An avoided level-crossing between the ground and an excited state in a **finite lattice** could become progressively sharper as the lattice size increases, leading to a **nonanalyticity** at $g = g_c$ in the **infinite lattice limit**.

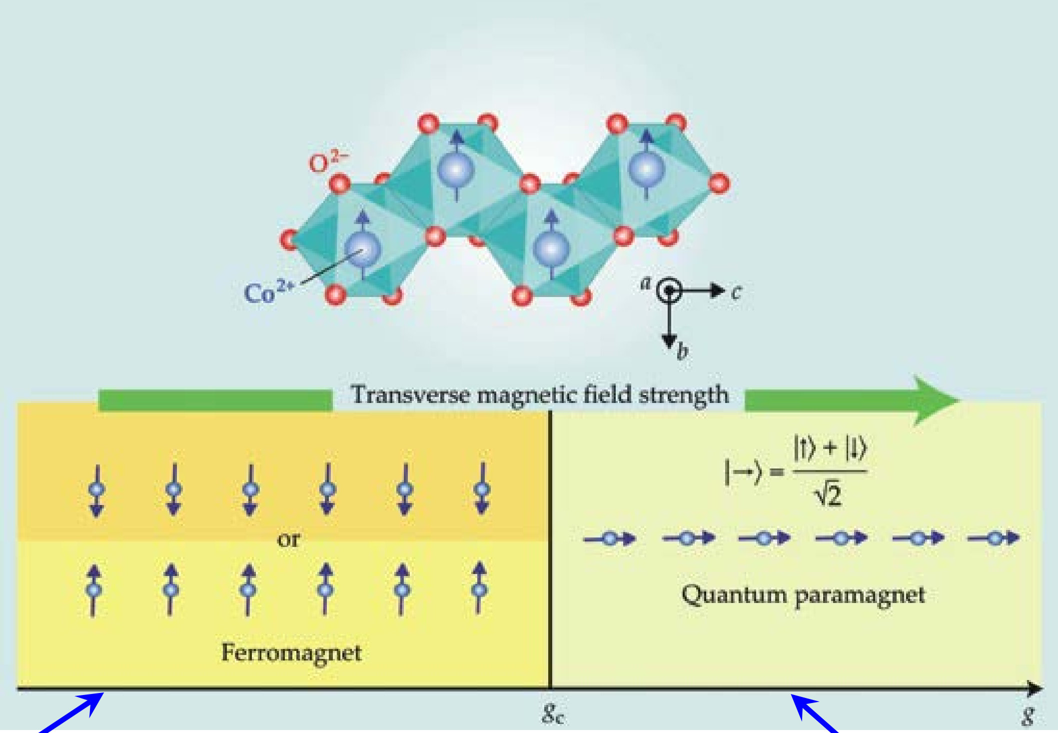
DEFINITION: Any point of nonanalyticity in the ground state energy of the infinite lattice system signifies quantum phase transition.

❑the nonanalyticity could be either the limiting case of an avoided level-crossing or an actual level-crossing.

❑QPT is usually accompanied by a qualitative change in the nature of the correlations in the ground state as one changes parameter in the Hamiltonian

Simple Theoretical Model I: Quantum Criticality in Quantum Ising Chain ($CoNb_3O_3$)

Phys. Today **64(2)**, 29 (2011)



Quantum Ising chain in the transverse external magnetic field

$$\hat{H} = -J\sum_i \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z - g\sum_i \hat{\sigma}_i^x$$

$$J > 0, g > 0$$

Each ion has two possible states:

$$\hat{\sigma}_z|\uparrow\rangle = +|\uparrow\rangle \qquad \hat{\sigma}_z|\downarrow\rangle = -|\downarrow\rangle$$

The first term in the Hamiltonian prefers that the spins on neighboring ions are parallel to each other, whereas the second allows quantum tunneling between the $|\uparrow\rangle$ and $|\downarrow\rangle$ states with amplitude proportional to g

$$|0\rangle = |\uparrow\rangle_1 \otimes |\uparrow\rangle_2 \cdots \otimes |\uparrow\rangle_N$$

or

$$|0\rangle = |\downarrow\rangle_1 \otimes |\downarrow\rangle_2 \cdots \otimes |\downarrow\rangle_N$$

$$\lim_{|x_i - x_j| \to \infty} \langle 0|\hat{\sigma}_i^z \hat{\sigma}_j^z|0\rangle = N_0^2$$

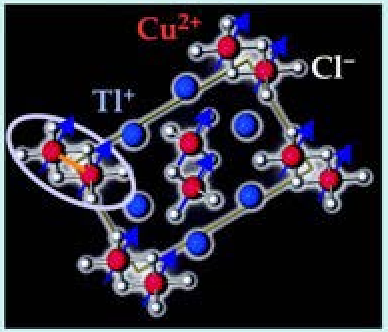
$$|0\rangle = |\rightarrow\rangle_1 \otimes |\rightarrow\rangle_2 \cdots \otimes |\rightarrow\rangle_N$$

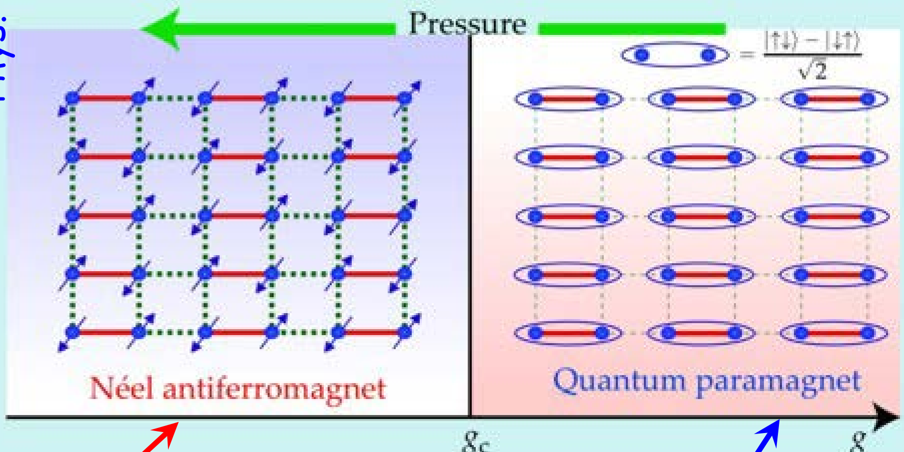
$$|\rightarrow\rangle_i = (|\uparrow\rangle_i + |\downarrow\rangle_i)/\sqrt{2}$$

$$\langle 0|\hat{\sigma}_i^z \hat{\sigma}_j^z|0\rangle \sim e^{-|x_i - x_j|/\xi}$$

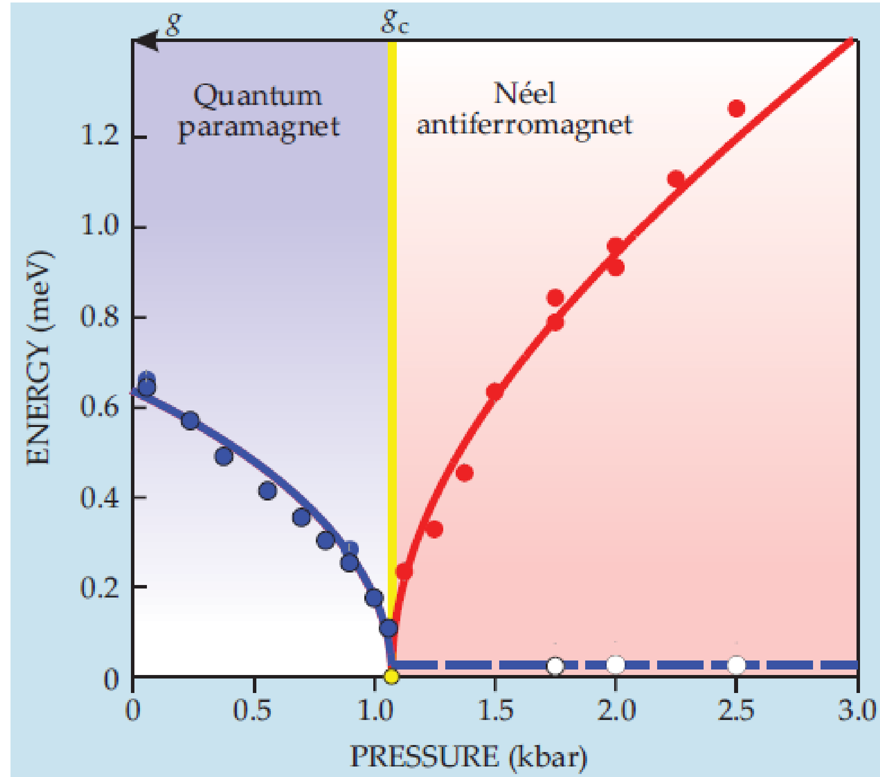
Simple Theoretical Model II: Quantum Criticality in Dimer Antiferromagnet ($TiCuCl_3$)

Phys. Today **64(2)**, 29 (2011)





Experiment: Phys. Rev. Lett. **100**, 205701 (2008)



❑ The two noncritical ground states of the dimer antiferromagnet have very different excitation spectra:

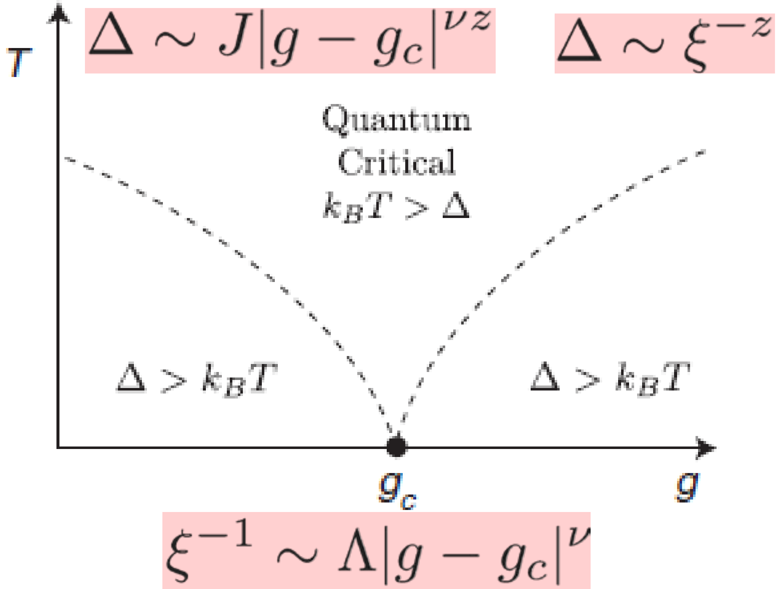
spin waves with nearly zero energy and oscillations of the magnitude of local magnetization

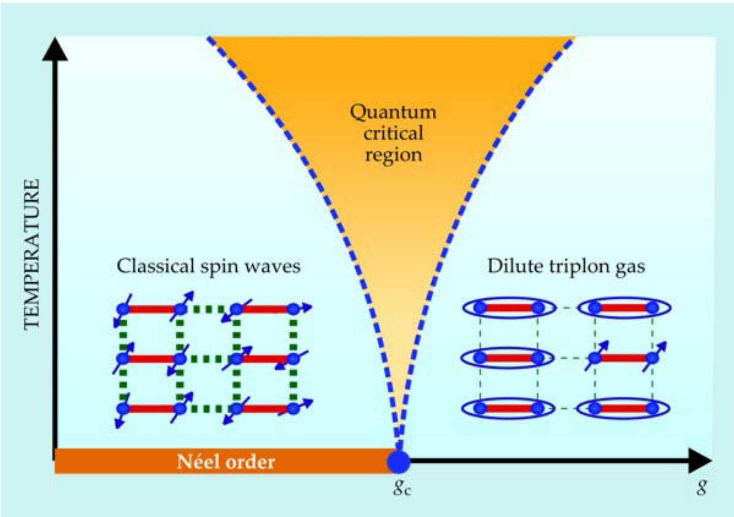
$$|0\rangle = \prod_i \otimes |s\rangle_i \qquad |s\rangle_i = (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2} \qquad |t_1\rangle_i = |\uparrow\uparrow\rangle$$

$$|t_0\rangle_i = (|\uparrow\downarrow\rangle + |\uparrow\downarrow\rangle)/\sqrt{2}$$

$$|t_m(\mathbf{k})\rangle = \frac{1}{\sqrt{N_d}} \sum_i e^{i\mathbf{k}\cdot\mathbf{r}_i} |t_m\rangle_i \prod_{j\neq i} \otimes |s\rangle_j \qquad |t_{-1}\rangle_i = |\downarrow\downarrow\rangle$$

How Quantum Criticality Extends to Non-Zero Temperatures





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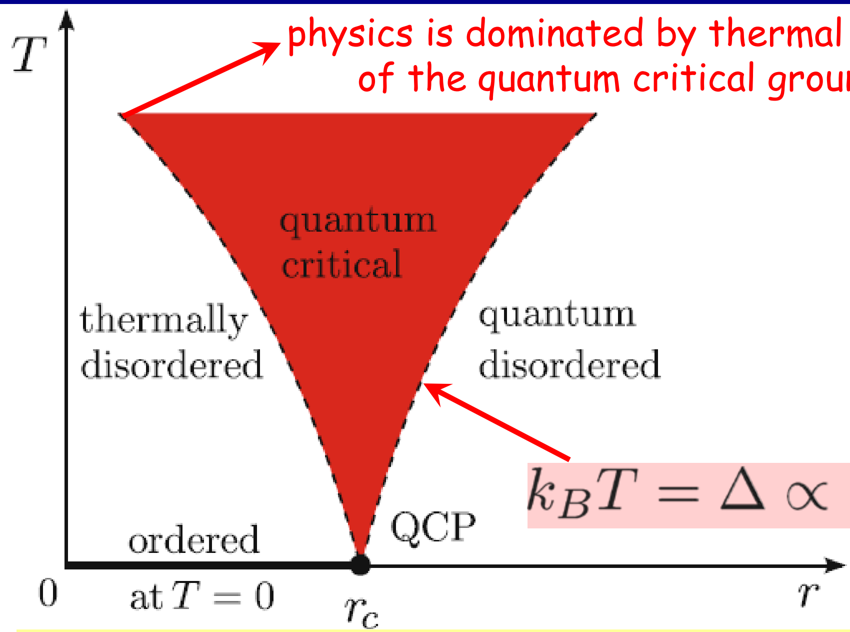
❑ **For small *g***, thermal effects induce spin waves that distort the Néel antiferromagnetic ordering. **For large *g***, thermal fluctuations break dimers in the blue region and form quasiparticles called triplons. The dynamics of both types of excitations can be described quasi-classically.

❑Quantum criticality appears in the intermediate orange region, where there is no description of the dynamics in terms of either classical particles or waves. Instead, the system exhibits the strongly coupled dynamics of nontrivial entangled quantum excitations of the quantum critical point g_c.

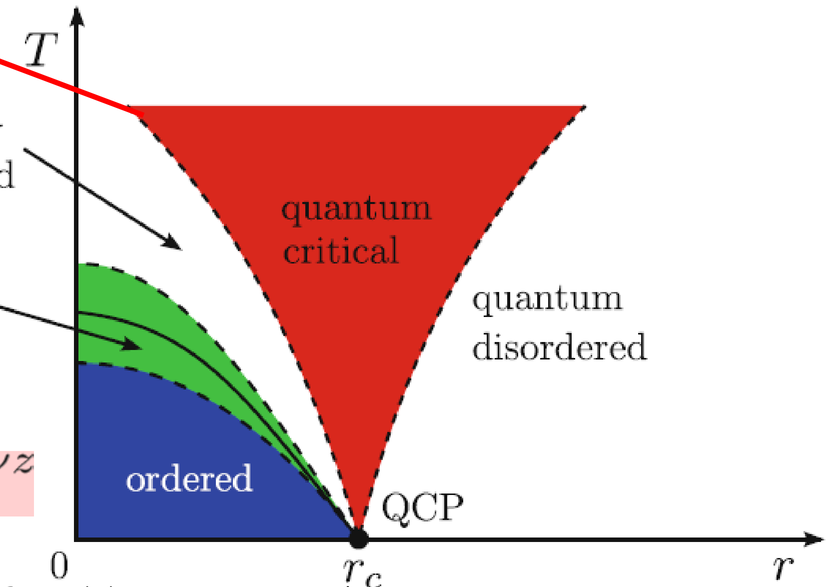
❑Wavefunction at $g=g_c$ is a **complex superposition of an exponentially large** set of configurations fluctuating at all length scales → thus, it cannot be written down explicitly due to long-range quantum entanglement which emerges for a very large number of electrons and between electrons separated at all length scales.

Scaling in the Vicinity of Quantum Critical Points

physics is dominated by thermal excitations of the quantum critical ground state



$$k_B T = \Delta \propto |r - r_c|^{\nu z}$$



$$Z = \int dp\,dq\, e^{-\beta H(p,q)} = \int dp\, e^{-\beta K(p)} \int dq\, e^{-\beta U(q)} \sim \int dq\, e^{-\beta U(q)}$$

$$Z = \mathrm{Tr}\, e^{-\beta \hat{H}} = \lim_{N\to\infty} \left(e^{-\beta \hat{T}/N} e^{-\beta \hat{U}/N} \right)^N$$

statics and dynamics coupled in quantum partition function $\Rightarrow$

QPT in d dimensions is equivalent to classical transition in higher d+z dimensions

$$Z = \int \mathcal{D}[\Phi] \exp\left\{ -\int_0^{1/T} d\tau \int d^D\mathbf{r}\, \mathcal{L}[\Phi(\mathbf{r},\tau)] \right\}$$

$\Phi(\mathbf{r},\tau)$ represents fluctuations of the order parameter and it depends on the imaginary time τ=-iβ/ħ which takes values in the interval [0,1/T]; the imaginary time direction acts like an extra dimension, which becomes infinite for T→0

CAVEATS:
- mapping holds for thermodynamics only
- resulting classical system can be unusual and anisotropic
- extra complications without classical counterpart may arise, such as Berry phases

$$T = 0: \quad f_{\mathrm{sing}}(G,h) = b^{-(D+z)} f_{\mathrm{sing}}(b^{y_g}G, b^{y_h})$$

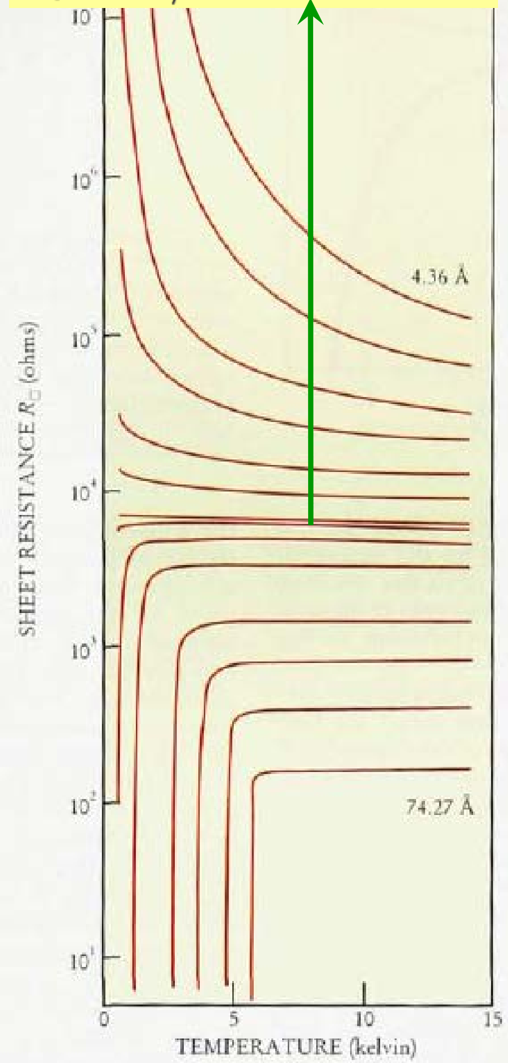
$$G = |g - g_c|/g_c$$

$$T > 0: \quad f_{\mathrm{sing}}(G,h,T) = b^{-(D+z)} f_{\mathrm{sing}}(b^{y_g}G, b^{y_h}, b^z T)$$

$$1/y_g = \nu$$

Example of Scaling Analysis: Superconductor-Insulator QPT in Thin Films

$R_c = h/4e^2 = 6450\ \Omega$



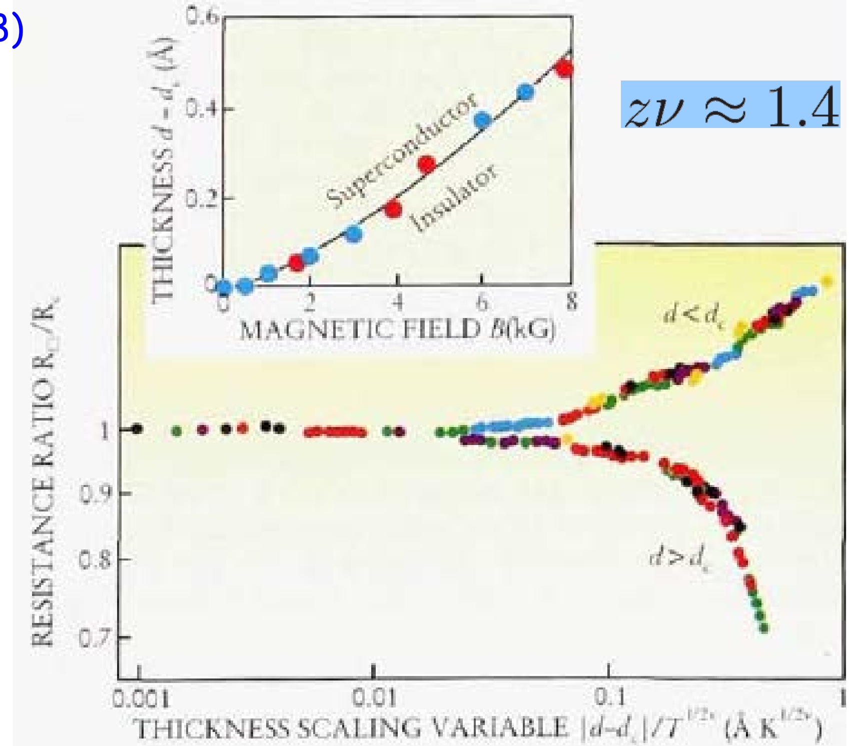
Phys. Today **51(11)**, 39 (1998)

The success of finite-size scaling analyses of the superconductor-insulator transitions as a function of film thickness or applied magnetic field provides strong evidence that $T = 0$ quantum phase transitions are occurring

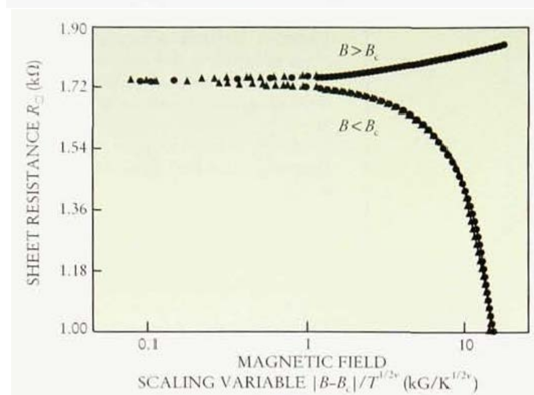
$\xi \sim g^{-\nu}$ $\xi_\tau \sim \xi^z$

$$R_\square = R_c f\left(\frac{g}{T^{1/z\nu}}\right)$$

$g = B$ or $g = d$

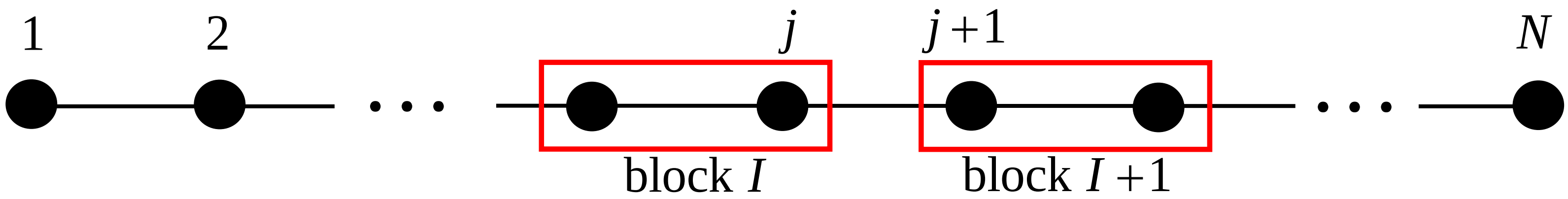


$z\nu \approx 1.4$



$z\nu \approx 1.36$

RG Coarse Graining for 1D Quantum Ising Model in Transverse Magnetic Field



$$\hat{H} = \sum_{j\in\text{odd}} \hat{H}_{\text{block},j} + \sum_{j\in\text{even}} \hat{H}_{\text{coupling},j} \qquad \hat{H}_{\substack{\text{block}\\ \text{coupling}},j} = -J\hat{\sigma}_j^z\hat{\sigma}_j^z - h\hat{\sigma}_j^x$$

$$\hat{H}_{\text{block},j} = \begin{array}{c} \begin{array}{cccc} |\uparrow\uparrow\rangle & |\uparrow\downarrow\rangle & |\downarrow\uparrow\rangle & |\downarrow\downarrow\rangle \end{array} \\ \begin{pmatrix} -J & 0 & -h & 0 \\ 0 & J & 0 & -h \\ -h & 0 & J & 0 \\ 0 & -h & 0 & -J \end{pmatrix} \end{array} \left\{ \begin{array}{l} |1\rangle \\ |2\rangle \\ |3\rangle \\ |4\rangle \end{array} \right. = \left\{ \begin{array}{l} -\sqrt{J^2+h^2}\,|1\rangle \\ -\sqrt{J^2+h^2}\,|2\rangle \\ \sqrt{J^2+h^2}\,|3\rangle \\ \sqrt{J^2+h^2}\,|4\rangle \end{array} \right.$$

keep $|1\rangle$ and $|2\rangle$ only

$$\hat{P}_I = |1\rangle\langle 1| + |2\rangle\langle 2|$$

$$\tilde{\sigma}_z = |1\rangle\langle 1| - |2\rangle\langle 2|$$

$$\tilde{\sigma}_x = |1\rangle\langle 2| + |2\rangle\langle 1|$$

$$\hat{P} = \hat{P}_1 \otimes \hat{P}_2 \otimes \cdots \hat{P}_{N/2}$$

projector onto coarse-grained system

$$\hat{P}_I \hat{H}_{\text{block},j} \hat{P}_I = -\sqrt{J^2+h^2}\,\hat{P}_I$$

$$\tilde{\sigma}_I^z \qquad \frac{J}{\sqrt{J^2+h^2}}\tilde{\sigma}_{I+1}^z \qquad \frac{h}{\sqrt{J^2+h^2}}\tilde{\sigma}_I^x$$

$$\left(\hat{P}_I \otimes \hat{P}_{I+1}\right)\hat{H}_{\text{coupling},j}\left(\hat{P}_I \otimes \hat{P}_{I+1}\right) = -J\left(\hat{P}_I\hat{\sigma}_z^j\hat{P}_I\right)\otimes\left(\hat{P}_{I+1}\hat{\sigma}_z^{j+1}\hat{P}_{I+1}\right) - h\left(\hat{P}_I\hat{\sigma}_x^j\hat{P}_I\right)\otimes\hat{P}_{I+1}$$

RG Flow for 1D Quantum Ising Model in Transverse Magnetic Field

$$\hat{H} = -J\sum_{j=1}^{N}\hat{\sigma}_j^z\hat{\sigma}_{j+1}^z - h\sum_{j=1}^{N}\hat{\sigma}_j^x$$

original Hamiltonian for N spins

$$\tilde{H} = \hat{P}\hat{H}\hat{P} = -\sqrt{J^2+h^2}\sum_{I=1}^{N/2}\tilde{P}_I - \frac{J^2}{\sqrt{J^2+h^2}}\sum_{I=1}^{N/2-1}\tilde{\sigma}_I^z\tilde{\sigma}_{I+1}^z - \frac{h^2}{\sqrt{J^2+h^2}}\sum_{I=1}^{N/2}\tilde{\sigma}_I^x$$

coarse grained Hamiltonian for N/2 spins after one RG iteration

RG flow equations

$$\begin{pmatrix} J' \\ h' \end{pmatrix} = \begin{pmatrix} \dfrac{J^2}{\sqrt{J^2+h^2}} \\ \dfrac{J^2}{\sqrt{J^2+h^2}} \end{pmatrix} \Leftrightarrow K' = \frac{h'}{J'} = \left(\frac{h}{J}\right)^2 = K^2$$

$$K^* = \left(K^*\right)^2 \overset{\text{fixed points}}{\Rightarrow} \begin{cases} K^* = 0 & \text{stable} \to \text{ferromagnet} \\ K^* = \infty & \text{stable} \to \text{paramagnet} \\ K^* = K_c = 1 & \text{unstable} \to \text{criticality} \end{cases}$$

linearize RG equations close to K_c

$$K' - K_c \approx \left.\frac{dK'}{dK}\right|_{K_c}\left(K - K_c\right)$$

$$\lambda_K = \left.\frac{dK'}{dK}\right|_{K_c} = 2 = b^{y_K} \Rightarrow y_K = 1$$

extract critical exponents from RG flow

$$\xi(K') = \frac{1}{b}\xi(K) \Rightarrow \xi = \frac{\text{const}}{K - K_c} \propto (K - K_c)^{-\nu} \Rightarrow \nu = \frac{1}{y_K} = 1$$

$$\left\langle \hat{\sigma}_z^1 \right\rangle = \frac{J}{\sqrt{J^2+h^2}}\left\langle \tilde{\sigma}_z^1 \right\rangle = \frac{1}{\sqrt{1+K^2}}\left\langle \tilde{\sigma}_z^1 \right\rangle$$

$$m(K) = b^{-x}m(b^{y_K}K) \Rightarrow b^{-x} = \frac{1}{\sqrt{1+K_c^2}} \Rightarrow x = \frac{1}{2},\ \beta = 1 - x = \frac{1}{2}$$