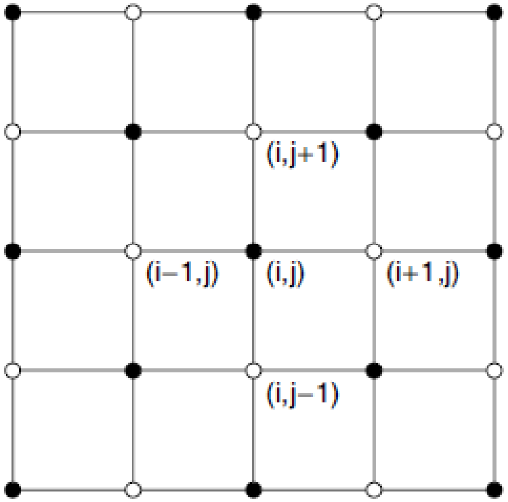
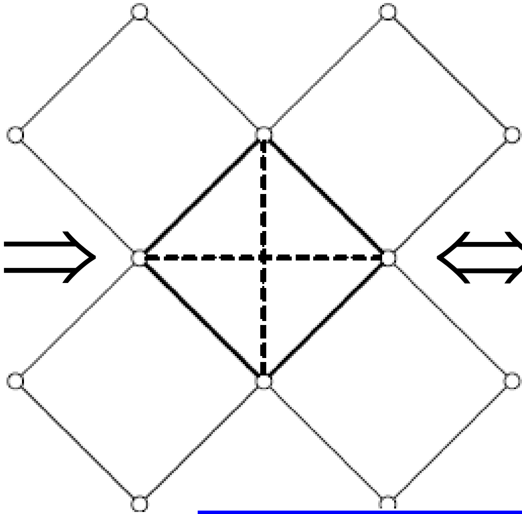
Scaling Hypothesis and Critical Exponents Extracted from RG



$\Rightarrow$



$\Leftrightarrow$

$$K_1' = \tfrac{1}{4}\ln(\cosh 4K)\,,$$

$$K_2' = \tfrac{1}{8}\ln(\cosh 4K)\,,$$

$$\cancel{K_3' = \tfrac{1}{8}\ln(\cosh 4K) - \tfrac{1}{2}\ln(\cosh 2K)}$$

0 [or $K_c^3(K') = -0.05323$]

$$\boxed{\begin{aligned} K' &= K_1' + K_2' \\ K' &= \tfrac{3}{8}\ln(\cosh 4K) \end{aligned}}$$

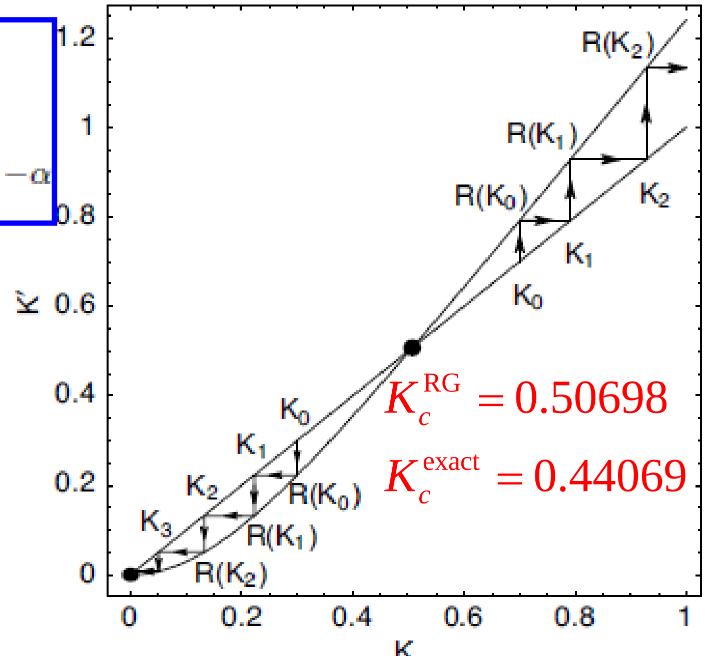
critical exponents from RG flow:

$$\boxed{\begin{aligned} C_V &= -T\left(\frac{\partial^2 F}{\partial T^2}\right)_V \\ C_V &\sim |T - T_c|^{-\alpha} \sim |t|^{-\alpha} \end{aligned}}$$

$$\ln \mathcal{Z} = N f(K)$$

$$f(K') = 2f(K) - \ln\left[2(\cosh 4K)^{1/8}(\cosh 2K)^{1/2}\right]$$

removal of short-range degrees of freedom
results in an expression that is analytic in K



$K_c^{RG} = 0.50698$

$K_c^{exact} = 0.44069$

focus on the singular part of free energy

$$f_s(K) = b^{-d} f_s(K')\,; \qquad b = \sqrt{2} \text{ and } d = 2$$

$$K' = R(K) = K_c + \left.\frac{dR}{dK}\right|_{K=K_c}(K - K_c) + \cdots$$

linearize recursion
in RG flow

$$\delta K' = \lambda\,\delta K = b^y\,\delta K\,; \quad \delta K' = K' - K_c \text{ and } \delta K = K - K_c$$

$$\lambda = R'|_{K_c};\ \lambda(b)\lambda(b) = \lambda(b^2) \Rightarrow \lambda = b^y \qquad\qquad b = |t|^{1/y}$$

$$f_s(\delta K) = b^{-d} f_s(b^y\,\delta K); f_s(t) = b^{-d} f_s(b^y t); f_s(t) = |t|^{d/y} f_s(t/|t|)$$

$$f_s \sim |t|^{2-\alpha},\ d/y = 2 - \alpha$$

$$f_s(t) = |t|^{2-\alpha} f_s(t/|t|)$$

$$y = \frac{\ln\lambda}{\ln b} = \frac{1}{\ln b}\ln\left(\left.\frac{dR}{dK}\right|_{K=K_c}\right) = \frac{\ln(\frac{3}{2}\tanh 4K_c)}{\ln b}$$

α=0.131

General RG Algorithm for Critical Phenomena
(or "Problems in Physics with Many Scales of Length")

1. Critical exponents associated with a fixed point are calculated by linearizing the RG recursion relations about that fixed point. Where there are recursion relations for n quantities,

$$K_i' = R_i(K_1, K_2, \ldots, K_n)\,,$$

for $i = 1, 2, \ldots, n$, linearization yields

$$\delta K_i' = \sum_{j=1}^{n} \frac{\partial R_i}{\partial K_j}\bigg|_{\{K_i\}=\{K_i^*\}} \delta K_j \equiv \sum_{j=1}^{n} M_{ij}\delta K_j\,.$$

The eigenvectors and eigenvectors of the matrix M with entries M_{ij}, $\mathsf{M}U_i = \lambda_i U_i = b^{y_i}U_i$, yield the linear **scaling fields** U_i, for $i = 1, 2, \ldots, n$ in terms of which the singular part of the free energy is expressed as

$$f_s(U_1, U_2, \ldots, U_n) = b^{-d} f_s(b^{y_1}U_1, b^{y_2}U_2, \ldots, b^{y_n}U_n)\,. \qquad (1)$$

2. The form of the singular part of the free energy in Eq. (1) is a generalized homogeneous function

3. The exponents y_i in Eq. (1) determine the behavior of the U_i under the repeated action of the linear RG recursion relations. If $y_i > 0$, U_i is called **relevant**. If $y_i < 0$, U_i is called **irrelevant** and, if $y_i = 0$, U_i is called **marginal**. We see that if a relevant scaling field is non-zero initially, then the linear recursion relations will transform this quantity *away* from the critical point. Alternatively, an irrelevant scaling field will transform this quantity *toward* the critical point, while a marginal variable will be left invariant. Thus, relevant quantities must vanish at a critical point. For the 2D Ising model, the reduced temperature t is clearly a relevant scaling field, as the calculation in Eq. (8.21) demonstrates explicitly. The magnetic field must also vanish at the critical point, so this is another relevant variable.

4. The existence of the relevant, irrelevant, and marginal variables explains the observation of universality, namely, that ostensibly disparate systems (e.g. fluids and magnets) show the same critical behavior near a second-order phase transition,, including the same exponents for analogous physical quantities. In the RG picture, critical behavior is described entirely in terms of the *relevant* variables, while the microscopic differences between systems is due to *irrelevant* variables.