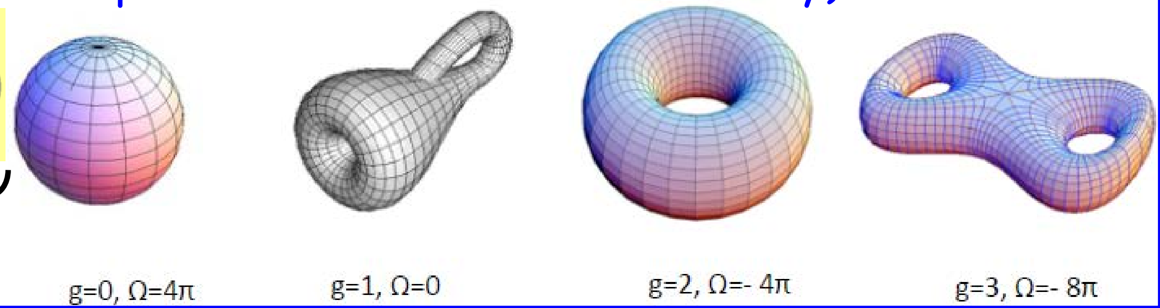
Crash Course on Topological Phases of Quantum Matter

Gauss-Bonnet theorem (applied to compact surfaces without boundary):

$$\underbrace{\frac{1}{4\pi}\int \kappa dA = \frac{\Omega}{4\pi}}_{\text{differential geometry}} = \underbrace{(1-g)}_{\text{topology}}$$



Differential geometry of wavefunctions:

$$\Psi_n(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} u_{n\mathbf{k}}(\mathbf{r})$$

Brillouin zone plays the role of the "surface"

$$\begin{aligned} \mathcal{A}_{n\mathbf{k}} &= \langle u_{n\mathbf{k}}| -i\nabla_{\mathbf{k}} |u_{n\mathbf{k}}\rangle \\ \phi_n^{\text{Berry}} &= \oint_{\partial\text{BZ}} \mathcal{A}_{n\mathbf{k}} \cdot d\mathbf{k} \\ \mathcal{F}_{n\mathbf{k}} &= \nabla \times \mathcal{A}_{n\mathbf{k}} \end{aligned}$$

"Berry connection" defines "Berry curvature" which replaces Gaussian curvature

$$\sigma_{xy} = \frac{e^2}{\hbar}\sum_n \int_{\text{BZ}} \frac{d^2k}{2\pi} f(\varepsilon_{n\mathbf{k}})\mathcal{F}_{n\mathbf{k}} \cdot \mathbf{e}_z = \frac{e^2}{h} N_{\text{Chern}}$$

Bulk-boundary correspondence: Identify Chern number N_{Chern} with the number of conducting edge states

Topological phases of matter: