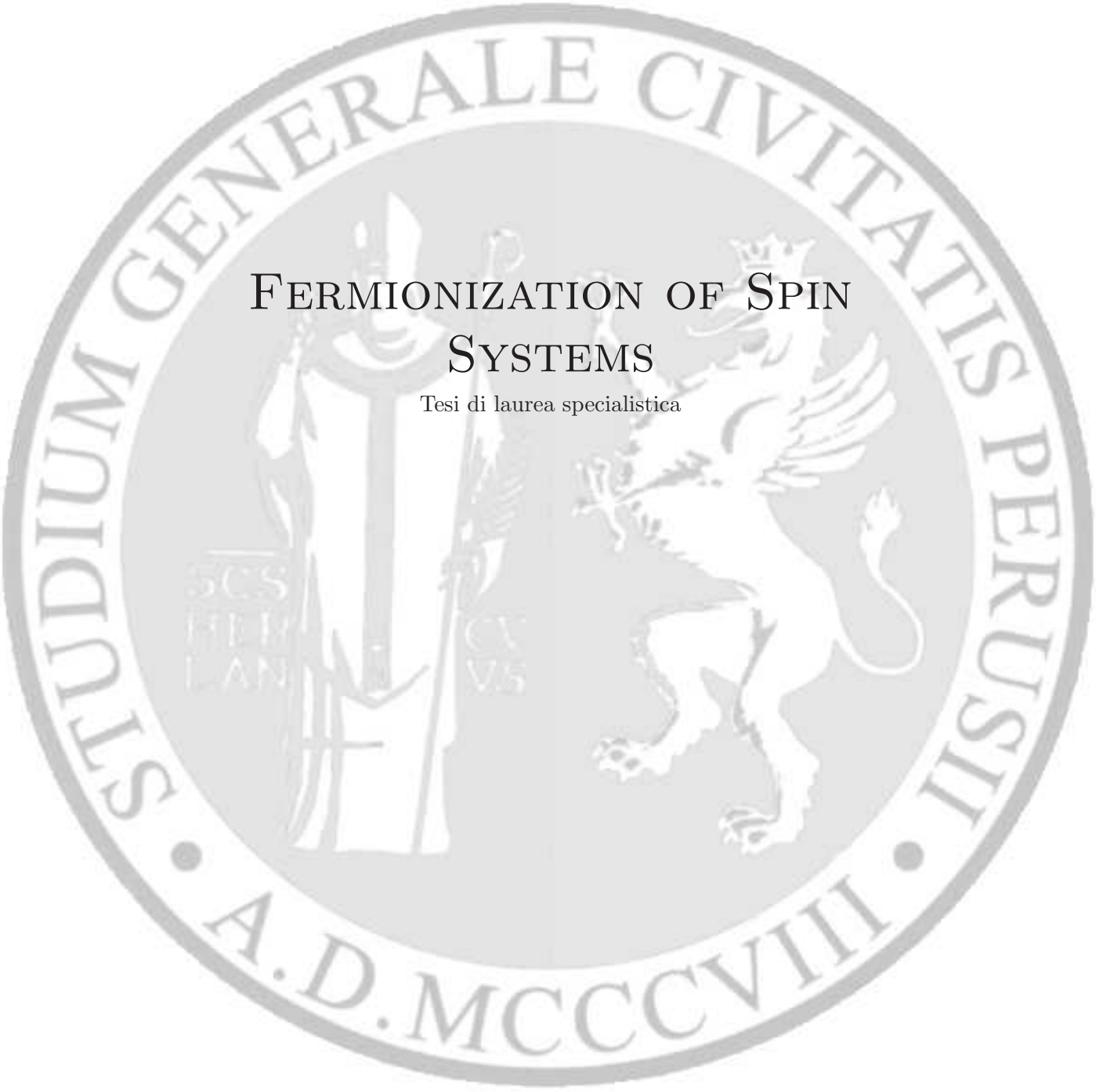
Matteo Mancini



Fermionization of Spin Systems

Tesi di laurea specialistica

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Fermionization of Spin Systems

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Introduction

Spin models are of crucial importance in physics.
Beside describing physical systems such as magnets, they are a place where to found exact solutions for non-trivial problems useful in all the sectors of physics. An extremely important example is the Bethe ansatz solution for an antiferromagnetic spin chain [1].

Thus it is important to have methods for finding exact solutions for spin models and this is the main purpose of this work.

The fundamental characteristic of spin models is the non-linearity, in fact even for apparently simple models as Heisenberg model, for which the Hamiltonian is quadratic in spins, is not as simple as quadratic Hamiltonian for fermions or bosons.

The non-linearity comes from the commutation relation of spin operators: commutators of spin operators are not c-numbers, but are still operators.

Traditional approaches to these models are classical. It means that spins are considered as fluctuating arrows around a fixed reference frame. For example ferromagnetic or antiferromagnetic systems have a preferential direction, the one of the magnetization, and spins fluctuates around it. This approach is called spin-wave approximation and is based on the existence of some global order and on the smallness of the fluctuations.

But spin-wave approximation is not always useful. For example it cannot be used when there isn't any global order due to the strong fluctuations of spins [2].

The sources of the disorder are various, but they all depend from topology. Indeed an important cause is the dimensionality, as low dimensions systems are always more frustrated, due to the absence of degrees of freedom.

Another source is geometry frustration, indeed simple square lattice or other Bravais lattice are not frustrated, but if we have, for example, triangular, pyramidal or hexagonal elements, these create frustration and, for example, the antiferromagnetic order is unfavorable. Classical spins may always escape frustration, as they can rotate freely, but quantum spins cannot, especially spin 1/2. Furthermore spin 1/2 has also strong quantum fluctuations, due to the smallness value of the spin.

In this work we propose non-semiclassical approaches to resolve spin models, to simplify them and, eventually, obtain exact solutions.

Our methods is based on the so called *"fermionization"*. As the name tell, fermionization is a procedure that mutates spin operators into fermion operators.

The first question might be: *why not bosons?*

In fact primal approaches converted spins into bosons, because spins in different sites obey to commutation relations.

These methods are useful in the limit of $S \to \infty$ [2,3], in that case projections of spin are near to be continuous and also the semiclassical approximation is applicable.

But spin 1/2 has only two projections. Thus, if we want to represent it with bosons we must introduce a strong repulsion among particles in the same site. This is not destructive for large dimensionality, because the probability for bosons to be in the same site is small; but for one or two dimensions it is a strong condition.

For $S = 1/2$ and low dimensionality it is better to represent spins as fermions.

Fermionization is very useful to diagonalize various problems. For example it has been used to exactly resolve the quantum XY model in one dimension [4,3] or to obtain an approximate solutions for the quantum antiferromagnetic Heisenberg chain [5,6], that in some cases may be more useful than the exact Bethe ansatz solutions because the wave functions are simpler. It has also been used to resolve the two dimensional antiferromagnetic Heisenberg model in various lattices configurations [5, 7, 8].

Recently fermionization has been used to describe the "spin fluid" state [9], i.e. a disordered state with strong correlations, proposed by P. W. Anderson [10], and it has also been applied to the Kondo lattice model [11, 12, 13, 14].

In this work we present the application of these methods to a very important model, spins placed on a two dimensional honeycomb lattice, that has been recently proposed by A. Kitaev [15].
This is a non-trivial model as it renders high anisotropy and high geometrical frustration, that comes mainly from the lattice structure, indeed it isn't a simple Bravais lattice, but it can be decomposed in two overlapping triangular sublattices.

Furthermore this model has an important integral of motion, that permits to solve it exactly.

We will apply two kinds of fermionization to this model and we will see that both render the same excitation spectrum. We will also see that, depending on the kind of information needed, the one is more useful than the other, because of the differences in the approaches.

Next we will go on to obtain some important results for the model, capitalizing on our formalism of fermionization. The main result will be the emergence of a Topological Quantum Order.

Indeed in this model the spin-spin correlations goes rapidly to zero with the distance for all energies, so, apparently, there is no order, because there is no local order parameter that can describe a phase transition. But, as we will see, the system undergoes various quantum phase transitions, so a new kind of order emerge, that has its origin in the topology.

Topological Order [16, 17] is a new theory proposed to describe systems that cannot be described by the Landau theory of phase transitions, indeed the latter is applicable in the case of a symmetry breaking and is driven by a local parameter, such as the magnetization in magnets.

Topological Quantum Order (TQO) can be described by non-local order parameter, such as ground state degeneracy or string order parameter.

Topological ordered systems have also been designed and studied in the context of quantum computation [18, 15], because of they are very robust to environmental noise and can escape decoherence.

The key feature, useful for Topological Quantum Computation, is the emergence of Anyons. Anyons are excitations typical of TQO systems in two dimensions. They are particles that obey a fractional statistics, i.e. they are not fermions nor bosons.
Some kinds of these anyons can be used to implement logical gates and quantum computation [19, 15].

In our honeycomb model Anyons will appear as vortex excitations, as happens also in p-wave superconductors [20, 21, 22, 23]. This similarity between the two models also emerge from the fermionization, in fact, as we will, see the honeycomb spin model is mapped to a p-wave BCS pairing Hamiltonian.

This work is organized as follows: in Chapter 1 we review the various fermionization methods, the Schwinger-Wigner representation, Jordan-Wigner representation and the recent Majorana representation.
For a better comprehension we will perform some classical application of the methods presented.
In addition in §1.3.2 we will suggest a new derivation for the Majorana representation proposed by A. Kitaev [15], connecting it to the Schwinger-Wigner fermionization.

Chapter 2 is devoted to the Kitaev's honeycomb model, after a presentation of its principal characteristics and symmetries, we will proceed with the fermionization to solve it.
We will explicitly diagonalize it for two configurations, the vortex-free and the vortex-lattice, using two kinds of fermionization.
For each configuration we will analyze the phase transitions between gapped and gapless phase. Then we will show the topological nature of phase transitions, by proving that spin-spin correlations in a general configuration is non-zero only between nearest neighbor.
Another important result will be the emergence of a BCS Hamiltonian that will be solved in the appendix A.

Chapter 1

Fermionization of spin 1/2

What follows is a review of various methods of fermionization.
They are connected each other, but at the same time each one has particular characteristics that make them useful for some applications.

1.1 Schwinger-Wigner representation

This first method was originally used to transform spins in bosons. That was useful for large value of spin or high dimensionality and resembles the results of spin wave theory.

Below we will describe the representation for both bosons and fermions. The Schwinger-Wigner representation is the following:

$$S^a = b^\dagger_\alpha \left(\frac{\sigma^a_{\alpha\beta}}{2} \right) b_\beta \tag{1.1}$$

where σ are Pauli matrix and b_1 e b_2 can be either fermions or bosons. Indeed we can verify that commutation rules for spins are satisfied by this representation, either if b are bosons or fermions. Explicitly the three components of spin are:

$$\begin{aligned} S^x &= \frac{1}{2}\left(b^\dagger_1 b_2 + b^\dagger_2 b_1 \right) \\ S^y &= \frac{i}{2}\left(b^\dagger_2 b_1 - b^\dagger_1 b_2 \right) \\ S^z &= \frac{1}{2}\left(b^\dagger_1 b_1 - b^\dagger_2 b_2 \right). \end{aligned}$$

We now verify the commutation relations:

$$\left[S^i, S^j\right] = i\varepsilon^{ijk} S^k$$

To this end we calculate the following commutators (upper sign is for bosons):

$$\begin{aligned}
\left[b_\alpha^\dagger b_\beta, b_\beta^\dagger b_\alpha\right] =& b_\alpha^\dagger b_\beta b_\beta^\dagger b_\alpha - b_\beta^\dagger b_\alpha b_\alpha^\dagger b_\beta = \\
=& b_\alpha^\dagger b_\alpha \pm b_\alpha^\dagger b_\alpha b_\beta^\dagger b_\beta - b_\beta^\dagger b_\beta \mp b_\alpha^\dagger b_\alpha b_\beta^\dagger b_\beta = b_\alpha^\dagger b_\alpha - b_\beta^\dagger b_\beta \\
\left[b_\alpha^\dagger b_\beta, b_\alpha^\dagger b_\alpha\right] =& b_\alpha^\dagger b_\beta b_\alpha^\dagger b_\alpha - b_\alpha^\dagger b_\alpha b_\alpha^\dagger b_\beta = \\
=& \left\{ \begin{array}{ll} b_\alpha^\dagger b_\alpha^\dagger b_\alpha b_\beta - b_\alpha^\dagger b_\beta - b_\alpha^\dagger b_\alpha^\dagger b_\alpha b_\beta & \text{(for bosons)} \\ \underbrace{b_\alpha^\dagger b_\alpha^\dagger}_{0} b_\alpha b_\beta - b_\alpha^\dagger b_\beta + b_\alpha^\dagger b_\alpha^\dagger b_\alpha b_\beta & \text{(for fermions)} \end{array} \right\} = -b_\alpha^\dagger b_\beta \\
\left[b_\alpha^\dagger b_\beta, b_\beta^\dagger b_\beta\right] =& b_\alpha^\dagger b_\beta
\end{aligned}$$

so the commutators:

$$\begin{aligned}
[S^x, S^y] &= \frac{i}{2}\left[b_1^\dagger b_2, b_2^\dagger b_1\right] = \frac{i}{2}\left(b_1^\dagger b_1 - b_2^\dagger b_2\right) = iS^z \\
[S^y, S^z] &= \frac{i}{4}\left[b_2^\dagger b_1 - b_1^\dagger b_2, b_1^\dagger b_1 - b_2^\dagger b_2\right] = \frac{i}{2}\left(b_2^\dagger b_1 + b_1^\dagger b_2\right) = iS^x \\
[S^z, S^x] &= \frac{1}{4}\left[b_1^\dagger b_1 - b_2^\dagger b_2, b_1^\dagger b_2 + b_2^\dagger b_1\right] = \frac{1}{2}\left(b_1^\dagger b_2 - b_2^\dagger b_1\right) = iS^y
\end{aligned}$$

The spin algebra spawn a $(2S+1)$-dimensional Hilbert space, but in the representation (1.1) the dimensionality is infinity for bosons, while for fermions is four. To shrink the space we have to impose $\boldsymbol{S}^2 = S(S+1)$. To do this we calculate:

$$\begin{aligned}
(S^x)^2 + (S^y)^2 &= \frac{1}{2}\left(b_1^\dagger b_2 b_2^\dagger b_1 + b_2^\dagger b_1 b_1^\dagger b_2\right) = \frac{1}{2}\left(b_1^\dagger b_1 + b_2^\dagger b_2 \pm 2 b_1^\dagger b_1 b_2^\dagger b_2\right) = \\
&= \frac{1}{2}\left(n_1 + n_2 \pm 2 n_1 n_2\right) \\
(S^z)^2 &= \frac{1}{4}\left(n_1^2 + n_2^2 - 2 n_1 n_2\right)
\end{aligned}$$

Thus:

$$\boldsymbol{S}^2 = (S^x)^2 + (S^y)^2 + (S^z)^2 =$$

$$= \begin{cases} \frac{1}{4}\left((n_1+n_2)^2 + 2(n_1+n_2)\right) = \left(\frac{n_1+n_2}{2}\right)\left(\left(\frac{n_1+n_2}{2}\right)+1\right) & \text{(for bosons)} \\ \frac{1}{4}\left(3n_1+3n_2-6n_1n_2\right) = \frac{3}{4}\left(n_1-n_2\right)^2 & \text{(for fermions)} \end{cases}$$

Where $n_\alpha = b_\alpha^\dagger b_\alpha$ and for fermions we have used $n_\alpha^2 = n_\alpha$.

From this results we note that using fermions we can only represent spin 1/2, which is indeed a special case.

To properly represents spin 1/2 we must impose this relation, valid either for fermions or bosons:

$$b_1^\dagger b_1 + b_2^\dagger b_2 = 1 \tag{1.2}$$

So this is equivalent to force a infinite repulsion among particles in the same site.

This constraint isn't as strong as it seems for high dimensions. Indeed the probability for two particles to occupy the same site decrease with the dimensionality, but for one or two dimension the constraint is significant and it needs to be taken into account. For example in a path integral formulation the constraint can be imposed by a Lagrange multiplier.

1.2 Jordan-Wigner representation

This representation is based on the anticommutation rule of Pauli matrix on the same site:

$$\left\{\sigma_i^a, \sigma_i^b\right\} = 2\delta^{ab} \qquad \left\{S_i^a, S_i^b\right\} = \frac{\delta^{ab}}{2} \tag{1.3}$$

If we write spin raising and lowering operators $S^+ = S^x + iS^y$, $S^- = S^x - iS^y$, these satisfies to the following anticommutation rule on the same site:

$$\left\{S_i^+, S_i^-\right\} = 1\,, \quad \left\{S_i^+, S_i^+\right\} = \left\{S_i^-, S_i^-\right\} = 0 \tag{1.4}$$

The analogy with fermionic creation and annihilation operators is manifest, so we define:

$$\psi_i^\dagger = S_i^+\,, \qquad \psi_i = S_i^- \tag{1.5}$$

To obtain S^z, we apply the commutation relation $[S^+, S^-] = 2S^z$, so:

$$S_i^z = \frac{1}{2}\left[\psi_i^\dagger, \psi_i\right] = \frac{1}{2}\left(\psi_i^\dagger\psi_i - \psi_i\psi_i^\dagger\right) = \psi_i^\dagger\psi_i - \frac{1}{2}$$

Unfortunately this is not so simple. In fact if we consider different sites, the spin behave as bosons not fermions, because they commute. Thus we have to introduce phase factors, so that spins on different sites commutes. These factors have to be non local. In general we write:

$$S_i^+ = \psi_i^\dagger U(i,\{j\}) \qquad S_i^- = U^\dagger(i,\{j\})\psi_i \qquad S_i^z = \psi_i^\dagger\psi_i - \frac{1}{2} \tag{1.6}$$

Where $U(i,\{j\})$ is a function of all the others ψ_j, that must satisfy $U^\dagger U = 1$ and $U^+ = U^{-1}$.

A general expression for $U(i)$ does not exist, but it can be derived depending on the dimensionality and on the specific problem. P. Jordan and E. Wigner [24] obtained for one dimension:

$$U_{1D}(i) = \prod_{k<i} e^{i\pi\psi_k^+\psi_k} \tag{1.7}$$

A possible expression for two dimensions was proposed recently by E. Fradkin [25]:

$$U_{2D}(i) = \prod_{k\neq i} e^{i\theta_{ik}\psi_k^+\psi_k} \tag{1.8}$$

where θ_{ik} is the angle that $k-i$ creates with any reference axis, whatsoever.

For two dimensions there are also other expressions. Indeed, differently from 1D, in two dimensions does not exist an ordination and so there are various possible representations. Which to choose depend on the problem, but it is not always possible to perform any trasformation.

However all the 2D representations restricted to 1D bring to the (1.7). Indeed if we consider the representation (1.8) in the one dimensional case the angle between sites is 0 for following sites, while it is π for previous sites.

Now we prove that the representation (1.8) permits the commutation between spins in different sites. We define the unitary operator:

$$u_k(i) = e^{i\theta_{ik}\psi_k^\dagger\psi_k} \qquad \text{so that} \quad U(i) = \prod_{k\neq i} u_k(i)$$

which satisfies the following relations:

$$\psi_k^\dagger u_k(j) = \psi_k^\dagger\left(1 + i\theta_{jk}\psi_k^\dagger\psi_k + \dots\right) = \psi_k^\dagger$$

$$\psi_k^\dagger u_k^\dagger(j) = \psi_k^\dagger$$
$$\psi_k u_k(j) = \psi_k e^{i\theta_{jk}} e^{-i\theta_{jk}\psi_k\psi_k^\dagger} = e^{i\theta_{jk}}\psi_k$$
$$\psi_k u_k^\dagger(j) = e^{-i\theta_{jk}}\psi_k$$

Using these equalities we prove the commutation of spins in different sites:

$$S_i^+ S_j^+ = S_j^+ S_i^+ \quad \text{with } i \neq j$$
$$S_i^+ S_j^+ = \psi_i^+ \prod_{k\neq i} u_k(i)\psi_j^+ \prod_{k\neq j} u_k(j) = \psi_i^+ \prod_{k\neq i,j} u_k(i)u_j(i)\psi_j^+ u_i(j) \prod_{k\neq j,i} u_k(j) =$$
$$= \psi_i^\dagger u_j(i)\psi_j^\dagger u_i(j) = \psi_i^\dagger e^{i\theta_{ji}}\psi_j^\dagger$$
$$S_j^+ S_i^+ = \psi_j^\dagger \prod_{k\neq j} u_k(j)\psi_i^\dagger \prod_{k\neq i} u_k(i) =$$

this is symmetric to the precedent, so we can write:

$$= \psi_j^\dagger e^{i\theta_{ij}}\psi_i^\dagger = -\psi_i^\dagger e^{i\theta_{ij}}\psi_j^\dagger$$

now using the fact that $\theta_{ij} = \theta_{ji} + \pi$, we have an extra minus sign that gives the correct result:

$$= \psi_i^\dagger e^{i\theta_{ji}}\psi_j^\dagger = S_i^+ S_j^+$$

Q.E.D.

Other relations can be obtained similarly.

As can be easy to understand, the non local string phase factor introduced by the transformation isn't always simple to treat.
Indeed for some applications of this transformation in two dimensions [25, 7, 8], the phase factors do not cancel each other, but they can be considered as a gauge field interacting with fermions, so the analysis is still possible.

Another possible representation in two dimensions is a generalization of the one dimensional one.
It is performed by considering a path that covers all the sites of the lattice and then applying the one dimensional transformation along that path.
If it is possible to do there will be no phase factors and the analysis can be simpler. We will return to this problem in the conclusions.

This last representation is the most convenient Jordan-Wigner representation and we will use it later in §2.3.

Also for the 1D case there is an equivalent representation that will be useful later. Actually it is the same as (1.7), but with different phase, and it is written in a way that may be simpler for computation. This transformation is performed by taking:

$$\begin{aligned} U_{1D}(i) &= \prod_{k<i} \sigma_k^z \\ \sigma_k^z &= 2c_k^\dagger c_k - 1 \end{aligned} \tag{1.9}$$

For which is still valid $U = U^\dagger$. Now we check up that this representation reproduces the commutation relation for spins. For the same site it must happens that:

$$\begin{aligned} \left[S_i^+, S_i^-\right] &= 2S_i^z \\ \left[S_i^+, S_i^-\right] &= \prod_{k<i} \sigma_k^z c_i^\dagger \prod_{k<i} \sigma_k^z c_i - \prod_{k<i} \sigma_k^z c_i \prod_{k<i} \sigma_k^z c_i^\dagger = \\ &= c_i^\dagger c_i - c_i c_i^\dagger = 2c_i^\dagger c_i - 1 = 2S_i^z \end{aligned}$$

While for different sites, with $i < j$:

$$\begin{aligned} \left[S_i^+, S_j^-\right] &= 0 \\ \left[S_i^+, S_j^-\right] &= \prod_{k<i} \sigma_k^z c_i^\dagger \prod_{k<j} \sigma_k^z c_j - \prod_{k<j} \sigma_k^z c_j \prod_{k<i} \sigma_k^z c_i^\dagger = \\ &= -\prod_{i \leqslant k<j} \sigma_k^z c_i^\dagger c_j - \prod_{i \leqslant k<j} \sigma_k^z c_j c_i^\dagger = 0 \end{aligned}$$

Where we have used the following equivalence:

$$c_i^\dagger \sigma_i^z = c_i^\dagger \left(2c_i^\dagger c_i - 1\right) = c_i^\dagger \left(1 - 2c_i c_i^\dagger\right) = \left(1 - 2c_i^\dagger c_i\right) c_i^\dagger = -\sigma_i^z c_i^\dagger$$
$$c_i^\dagger \sigma_i^z = -\sigma_i^z c_i^\dagger$$

1.2.1 Application to the Heisenberg model

To show the use of these transformations we now discuss the one-dimensional XXZ spin model following the discussion by I. Affleck [3]. Consider the

Hamiltonian:

$$H = -J \sum_j \left[S_j^x S_{j+1}^x + S_j^y S_{j+1}^y \right] - J_z \sum_j S_j^z S_{j+1}^z \tag{1.10}$$

Where j marks the sites of a one-dimensional lattice and $S^{x,y,z}$ are quantum spin 1/2. In the limit of $J_z = 0$ this reduces to the one-dimensional quantum XY-model, while for $J_z = \pm J$ we have the well-known isotropic Heisenberg model, which describes respectively a ferromagnet or an antiferromagnet.

The Heisenberg model can be solved in a semiclassical way that is valid in the limit of $S \to +\infty$, this method treat spins as classical vectors weakly oscillating around the direction of magnetization.
The JW-transformation is a non-semiclassical approach to spin system valid only for spin 1/2, which is indeed a special case.

Before applying the transformation to the model we rewrite the Hamiltonian (1.10) in terms of spin raising and lowering operator defined above

$$H = -\frac{J}{2} \sum_j \left[S_{j+1}^+ S_j^- + S_{j+1}^- S_j^+ \right] - J_z \sum_j S_j^z S_{j+1}^z \tag{1.11}$$

Now we perform the one-dimensional JW transformation (1.6) using the phase factor (1.7). Fermionization of the first term left out only a factor $e^{i\pi \psi_j^\dagger \psi_j} = e^{i\pi \hat{n}_j}$, so we have:

$$-\frac{J}{2} \sum_j S_{j+1}^+ S_j^- = -\frac{J}{2} \sum_j \psi_{j+1}^\dagger e^{i\pi \hat{n}_j} \psi_j = -\frac{J}{2} \sum_j \psi_{j+1}^\dagger \psi_j \tag{1.12}$$

where we have made use of the relation

$$e^{i\pi \hat{n}_j} \psi_j = \left[1 + \psi_j^\dagger \psi_j \sum_{k=1}^{+\infty} (i\pi)^k \right] \psi_j = \psi_j$$

Indeed the ψ_j operator, when acting on the right, destroy a fermionic state on the site j and we can put $\hat{n}_j = 0$, obtaining the same relation. Then the second term is

$$-\frac{J}{2} \sum_j S_{j+1}^- S_j^+ = -\frac{J}{2} \sum_j \psi_{j+1} e^{-i\pi \hat{n}_j} \psi_j^\dagger = \frac{J}{2} \sum_j \psi_{j+1} \psi_j^\dagger = -\frac{J}{2} \sum_j \psi_j^\dagger \psi_{j+1} \tag{1.13}$$

the minus sign in the third passage comes from the exponential $e^{i\pi}$, in fact

the operator $\psi_j^\dagger$ creates a fermionic state in the site j and so we can let $\hat{n}_j = 1$ in the exponential. Note that these first two terms are hopping terms for electron, with hopping probability equal to J.

The third term of the Hamiltonian becomes

$$- J_z \sum_j S_{j+1}^z S_j^z = -J_z \sum_j \left(\hat{n}_{j+1} - \frac{1}{2}\right)\left(\hat{n}_j - \frac{1}{2}\right) \tag{1.14}$$

that represent an interaction between electrons in adjacent sites.

So the fermionized Hamiltonian is, neglecting a constant term:

$$H_{XXZ} = -\frac{J}{2}\sum_j \left(\psi_{j+1}^\dagger \psi_j + \text{h.c.}\right) + J_z \sum_j n_j - J_z \sum_j n_{j+1} n_j \tag{1.15}$$

It is interesting the fact that the pure XY Hamiltonian, with $J_z = 0$, is equivalent to a non-interacting fermion problem.

Now we pass to momentum space, by Fourier transforming, since the Hamiltonian is traslational invariant, and we obtain:

$$\psi_j = \frac{1}{\sqrt{N}} \sum_q \psi_k e^{iqj} \tag{1.16}$$

$$\psi_j^\dagger = \frac{1}{\sqrt{N}} \sum_q \psi_q^\dagger e^{-iqj} = \frac{1}{\sqrt{N}} \sum_q \psi_{-q}^\dagger e^{iqj} \tag{1.17}$$

where we have put the lattice spacing $a = 1$, that can be recovered by simple dimensional analysis. N is the number of spin site of the chain.
The discrete momentum takes values on the first Brillouin zone:

$$k = \frac{2\pi}{N} n \qquad \text{with } n = 0, 1, \ldots, N-1$$

We can verify what represent $\psi_q^\dagger$ and ψ_q, by writing the inverse transform:

$$\psi_q = \frac{1}{\sqrt{N}} \sum_j \psi_j e^{-iqj} \tag{1.18}$$

$$\psi_q^\dagger = \frac{1}{\sqrt{N}} \sum_j \psi_j^\dagger e^{iqj} \tag{1.19}$$

and now calculating the anticommutators:

$$\left\{\psi_p, \psi_q^\dagger\right\} = \frac{1}{N}\sum_{jk} e^{-iqj+ipk} \underbrace{\left\{\psi_j, \psi_k^\dagger\right\}}_{\delta_{jk}} =$$
$$= \frac{1}{N}\sum_{j} e^{i(p-q)j} = \delta_{pq} \tag{1.20}$$

Thus these are creation and annihilation operators for fermion with definite momentum k.

Now let us transform the terms in the Hamiltonian. Quadratic terms becomes:

$$-\frac{J}{2}\sum_j \left(\psi_{j+1}^\dagger \psi_j + \text{h.c.}\right) =$$
$$= -\frac{J}{2N}\sum_{k,k'}\Bigg(\underbrace{\sum_j e^{i(k'-k)j}}_{N\delta_{k,k'}} e^{-ik}\psi_k^\dagger\psi_{k'} + \underbrace{\sum_j e^{-i(k'-k)j}}_{N\delta_{k,k'}} e^{ik'}\psi_k^\dagger\psi_{k'}\Bigg) =$$
$$= -J\sum_k cos(k)\psi_k^\dagger\psi_k$$

and

$$-J_z\sum_j n_j = -J_z\sum_j \psi_j^\dagger\psi_j = -J_z\sum_k \psi_k^\dagger\psi_k$$

The quartic term can be seen as an interaction term

$$\sum_{jk} V_{jk} n_j n_k$$

with potential $V_{jk} = V_{j-k} = -\frac{J_z}{2}$ for $|j-k| = 1$ and zero otherwise, his Fourier transform is:

$$v_q = \sum_r V_r e^{-iqr} = -\frac{J_z}{2}\left(e^{-iq} + e^{iq}\right) = -J_z\cos q$$
$$V_{j-k} = \frac{1}{N}\sum_q v_q e^{iq(j-k)}$$

So we can write the interaction term in second quantization

$$\sum_{jk} V_{jk} n_j n_k = \frac{1}{N} \sum_{k,k'} \sum_{q} v_q \psi^\dagger_{k'+q} \psi^\dagger_{k-q} \psi_k \psi_{k'} =$$
$$= -\frac{J_z}{N} \sum_{k,k',q} \cos(q)\, \psi^\dagger_{k'+q} \psi^\dagger_{k-q} \psi_k \psi_{k'}$$

The whole transformation so holds

$$H = \sum_k \omega_k \psi^\dagger_k \psi_k - \frac{J_z}{N} \sum_{k,k',q} \cos(q) \psi^\dagger_{k'+q} \psi^\dagger_{k-q} \psi_k \psi_{k'} \tag{1.21}$$

where

$$\omega_k = J_z - J\cos(k)$$

Now we explore the limiting XY case, i.e. $J_z = 0$, with Hamiltonian

$$H_{XY} = -\frac{J}{2} \sum_i \left[\psi^\dagger_{i+1} \psi_i + \text{h.v.} \right] \tag{1.22}$$

The interaction term now identically vanishes and this is a simple hopping Hamiltonian. We have the following spectrum (fig. 1.1) for the single fermion, recovering the lattice spacing a:

$$\omega_k = -J\cos(ka)$$

This spectrum is shown in fig. 1.1 and it has negative energy states , i.e. fermions with $k < |\frac{\pi}{2a}|$. We can define the ground state for this system as the state with all negative states filled

$$|\psi_g\rangle = \sum_{k<|\pi/2a|} \psi^\dagger_k |0\rangle$$

where $|0\rangle$ is the vacuum state for the fermions. So we have an half-filled band, as we can verify by explicit calculation

$$\langle n_j \rangle = \frac{1}{N} \sum_{k,q} e^{i(k'-k)j} \langle \psi^\dagger_{k+q} \psi_k \rangle$$

For the expectation value to be nonzero it must be $q = 0$, and for the ground

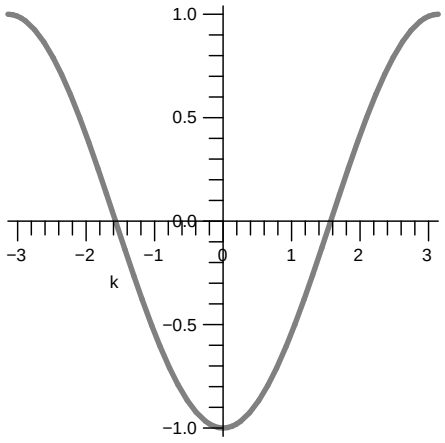


Figure 1.1: The spectrum of the XY model. It has negative energy states that are filled in the ground state

state we have $\langle\psi_k^\dagger\psi_k\rangle = 1$ for $k < |\frac{\pi}{2a}|$ and zero otherwise. So we have:

$$\langle n_j\rangle = \frac{1}{N}\sum_k \langle\psi_k^\dagger\psi_k\rangle = \frac{1}{N}\sum_{k<|\pi/2a|} 1$$

From the shape of the spectrum we can say that the sum gives a half of the sites available, i.e. $N/2$, as we can see passing to the continuum limit in momentum space

$$\sum_{k<|\pi/2a|} \to \frac{Na}{2\pi}\int_{-\pi/2a}^{\pi/2a} \tag{1.23}$$

$$\langle n_j\rangle = \frac{1}{N}\frac{Na}{2\pi}\frac{\pi}{a} = \frac{1}{2} \tag{1.24}$$

From this result we can calculate the average magnetization

$$\left\langle S_j^z\right\rangle = \langle n_j\rangle - \frac{1}{2} = 0 \tag{1.25}$$

Remarkably there is no average magnetization in the ground state of the XY model.

Excitation of the ground state can be made by adding a fermion with $k > |\frac{\pi}{2a}|$ or annihilating one with $k < |\frac{\pi}{2a}|$ to form a hole.

It is worth noting some characteristic of the spectrum.

First it is gapless, i.e. there are zero modes, corresponding to $k = \pm\frac{\pi}{2a}$, and so there are Goldstone modes, that marks the presence of long range correlation. Secondly the spectrum for low energy excitations is linear.
Low energy excitations are the responsible for long range correlations. So the only important states to study long range correlations are those close to the Fermi points, i.e. states with:

$$k = \pm k_F = \pm\frac{\pi}{2a}$$

thus we can linearize the single-particle spectrum $\omega_k = -J\cos(ka)$ in the vicinity of these points:

$$\begin{aligned} \omega_k &\approx -Ja(k + k_F)\ , \qquad \text{for } |k + k_F| \leqslant \Lambda \\ \omega_k &\approx Ja(k - k_F) \quad , \qquad \text{for } |k - k_F| \leqslant \Lambda \end{aligned}$$

where Λ is a cut-off which can be taken $\ll \frac{\pi}{2a}$. As we will see below this model with this approximations can be treated as a quantum field theory of free fermions with "speed of light" Ja. If we truncate the Fourier transform of the fermion operator to the cut-off, we can write:

$$\begin{aligned} \psi_j &= \frac{1}{\sqrt{N}} \sum_{|k|<\pi/a} e^{ikx_j}\psi(k) \\ &\approx \frac{1}{\sqrt{N}} \sum_{|q|\leqslant\Lambda} \left[e^{i\frac{\pi}{2a}x_j} e^{iqx_j}\psi\left(\frac{\pi}{2a} + q\right) + e^{-i\frac{\pi}{2a}x_j} e^{iqx_j}\psi\left(-\frac{\pi}{2a} + q\right)\right] \end{aligned}$$

Treating only long range correlation we can pass to the continuum limit and define the operators for left and right moving particles as

$$\begin{aligned} \psi_L(x) &= \frac{1}{\sqrt{Na}} \sum_{|q|\leqslant\Lambda} e^{ipx}\psi\left(\frac{\pi}{2a} + q\right) \\ \psi_R(x) &= \frac{1}{\sqrt{Na}} \sum_{|q|\leqslant\Lambda} e^{ipx}\psi\left(-\frac{\pi}{2a} + q\right) \end{aligned}$$

So we can write:

$$\psi_j \longrightarrow \sqrt{a}\left[(i)^j\psi_R(x) + (-i)^j\psi_L(x)\right]$$

Note that after the decomposition of a single lattice field we obtain two

continuous field in the limit $a \to 0$, this is a phenomenon called *fermion doubling* and it has origin from the fact that there are two regions for the low-energy limit.

Substituting into the XY Hamiltonian we have that due to the ortonormality of ψ_R and ψ_L the mixed terms goes to zero and we have:

$$\begin{aligned} H_{XY} =& -\frac{J}{2}\sum_j \left[\psi^\dagger_{j+1}\psi_j + h.c.\right] = \\ =& i\frac{Ja}{2}\sum_j \Big[\psi^\dagger_R(x)\psi_R(x+a) - \psi^\dagger_R(x+a)\psi_R(x) + \\ & -\psi^\dagger_L(x)\psi_L(x+a) + \psi^\dagger_L(x+a)\psi_L(x)\Big] \end{aligned}$$

In order to pass to the continuum we take first two terms,

$$\sum_j \left[\psi^\dagger_R(x)\psi_R(x+a) - \psi^\dagger_R(x+a)\psi_R(x)\right] =$$

expanding in series and then taking the limit $a \to 0$, we pass from the sum to the integral,

$$\approx \int dx \left[\psi^\dagger_R(x)\frac{d\psi_R}{dx}(x) - \frac{d\psi^\dagger_R}{dx}(x)\psi_R(x)\right] =$$

imposing Born-von Karman[1] boundary conditions, we can revert by parts,

$$= 2\int dx\, \psi^+_R(x)\frac{d\psi_R}{dx}(x)$$

The same passages can be done for ψ_L so the XY Hamiltonian becomes:

$$H_{XY} = iv\int dx \left[\psi^\dagger_R(x)\frac{d\psi_R}{dx}(x) - \psi^\dagger_L(x)\frac{d\psi_L}{dx}(x)\right] \tag{1.26}$$

where $v = Ja$ is the Fermi velocity.

Thus we obtain that the low-energy approximation of the XY model is equivalent to a Lorentz invariant massless Dirac fermion field theory, with "speed of light" v. Now we can read off all the long wave-length properties of the spin system from the Lorentz invariant field theory. From now on we

[1] Periodic boundary conditions

set $v = 1$. Consider for example the correlation function:

$$G^z(x,t) \equiv \langle S^z(x,t) S^z(0,0) \rangle \ .$$

Recalling

$$S^z_j = \psi^\dagger_j \psi_j - \frac{1}{2}$$

and

$$\left\langle \psi^\dagger \psi \right\rangle = \frac{1}{2}$$

we can write spin operator as a normal order product:

$$S^z_j =: \psi^\dagger_j \psi_j : \tag{1.27}$$

Written in terms of left and right moving fields, in the continuum limit, it separates in two part, one uniform and one alternating:

$$S^z(x) \approx a \left[\Big(: \psi^\dagger_L(x)\psi_L(x) : + : \psi^\dagger_R(x)\psi_R(x) : \Big) + \right. \\ \left. + (-1)^{\frac{x}{a}} \Big(\psi^\dagger_L(x)\psi_R(x) + \psi^\dagger_R(x)\psi_L(x) \Big) \right] \tag{1.28}$$

Where we have removed the normal ordering from the second part, because of $\left\langle \psi^\dagger_L \psi_R \right\rangle = 0$.

Thus we see that also the correlation function is separated in two parts:

$$\begin{aligned} G^z(x,t) \approx & \Big[\Big\langle : \psi^\dagger_L(x)\psi_L(x) :: \psi^\dagger_L(0)\psi_L(0) : \Big\rangle + \\ & + \Big\langle : \psi^\dagger_R(x)\psi_R(x) :: \psi^\dagger_R(0)\psi_R(0) : \Big\rangle \Big] + \\ & + (-1)^{\frac{x}{a}} \Big[\Big\langle \psi^\dagger_L(x)\psi \ \psi^\dagger_L(0)\psi_R(0) \Big\rangle + \\ & + \Big\langle \psi^\dagger_R(x)\psi_L(x)\psi^\dagger_R(0)\psi_L(0) \Big\rangle \Big] \end{aligned} \tag{1.29}$$

The separation in two parts is related to the "fermion doubling" and reflects the concurrence between ferromagnetic and antiferromagnetic properties of the model.

The calculation of the correlation function can be done using the Hamiltonian (1.26) and we obtain:

$$G^z(x,t) = -\left(\frac{1}{4\pi}\right)^2 \left[\left(\frac{1}{x^2_-} + \frac{1}{x^2_+} + (-1)^{\frac{x}{a}} \frac{2}{x_- x_+} \right) \right] , \tag{1.30}$$

where $x_\pm = (t \pm x)/2$. We note that both uniform and alternating pieces power decay as $1/x^2$.

This results are for the XY model, as we imposed $J_z = 0$. It is now interesting to see what happen if we take $J_z \neq 0$. In first approximation, i.e. neglecting the interaction of (1.21), we see that the spectrum remain gapless as we move from $J_z = 0$, this reflects the fact that the XY phase remain valid, with his long range correlations. This is a Kosterlitz-Thouless like phase transition.

The phase began to be gapful as we reach $J_z = 1$, i.e. the ferromagnetic point. At this point the system acquire a magnetization and the correlation decay exponentially.

1.3 Majorana representation

In this section we will explore some recent fermionic spin representations, involving Majorana fermions.
In nature it is not simple to have free Majorana fermions, because they are always recombined by the electromagnetic field to form complex fermions. But there can be screened systems where they could exists, this happens for example in p-wave superconductors [26].

Majorana fermions are particularly useful to define fermions on links, instead that on sites. This is necessary to describe disordered strong correlated systems [10] and in general topological systems [27], where we can found local order in the dual lattice, instead that in the real lattice.

Obviously there is a trivial, but very important, Majorana representation. It is the one that come directly from the Jordan-Wigner representation, when the fermion on each site is splitted in two Majorana.

The two representations that we will derive will have three and four Majorana fermions per site. Thus we will have unphysical states.

1.3.1 Standard Majorana representation

This representation was derived by A. Tsvelik [9,2] and P. Coleman [11], and was used to describe the "spin liquid" state and the Kondo lattice model.

We define the reality condition of Majorana fermion operators in this way:

$$\eta_i^+ = \eta_i \tag{1.31}$$

These fermions have the following anticommutation rule:

$$\left\{\eta_i, \eta_j^+\right\} = \{\eta_i, \eta_j\} = 2\delta_{ij} \tag{1.32}$$

thus:

$$\eta_i^2 = 1$$

We can represent spin operators by using three Majorana operators for each site, writing:

$$\boldsymbol{S}_i = -\frac{i}{4}\boldsymbol{\eta}_i \times \boldsymbol{\eta}_i \qquad S_i^a = -\frac{i}{4}\varepsilon_{abc}\eta_i^b\eta_i^c \tag{1.33}$$

Explicitly spin operators are:

$$\begin{aligned} S_i^x &= -\frac{i}{2}\eta_i^2\eta_i^3 \\ S_i^y &= -\frac{i}{2}\eta_i^3\eta_i^1 \\ S_i^z &= -\frac{i}{2}\eta_i^1\eta_i^2 \end{aligned} \tag{1.34}$$

We can verify the commutation rules for spins, for the same site:

$$\begin{aligned} [S^x, S^y] &= \frac{(i)^2}{4}\left(\eta^2\eta^3\eta^3\eta^1 - \eta^3\eta^1\eta^2\eta^3\right) = \frac{(i)^2}{2}\left(-\eta^1\eta^2\right) = iS^z \\ [S^y, S^z] &= \frac{(i)^2}{4}\left(\eta^3\eta^1\eta^1\eta^2 - \eta^1\eta^2\eta^3\eta^1\right) = \frac{(i)^2}{2}\left(-\eta^2\eta^3\right) = iS^x \\ [S^z, S^x] &= \frac{(i)^2}{4}\left(\eta^1\eta^2\eta^2\eta^3 - \eta^2\eta^3\eta^1\eta^2\right) = \frac{(i)^2}{2}\left(-\eta^3\eta^1\right) = iS^y \end{aligned}$$

It is easy also to verify that spins in different sites commute, indeed, being each spin composed by two Majorana, exchanging spins doesn't change the sign. We now verify that all states are physical, i.e. that $\boldsymbol{S}^2 = \frac{3}{4}$:

$$\begin{aligned} S^xS^x &= -\frac{1}{4}\eta^2\eta^3\eta^2\eta^3 = \frac{1}{4} \\ S^yS^y &= -\frac{1}{4}\eta^3\eta^1\eta^3\eta^1 = \frac{1}{4} \\ S^zS^z &= -\frac{1}{4}\eta^1\eta^2\eta^1\eta^2 = \frac{1}{4} \\ \boldsymbol{S}^2 &= \frac{3}{4} \end{aligned}$$

We now compute the dimensionality of the Fock space spanned by this representation. Consider a spin lattice composed by N sites. After the trans-

formation we'll have $3N$ different Majorana fermions, that we can combine two by two to create a complex fermion, indeed taken two Majorana a and b, we can define a complex fermion c in that way:

$$c^+ = a + ib \qquad c = a - ib$$
$$\left\{c, c^+\right\} = 1 \qquad \{c, c\} = \left\{c^+, c^+\right\} = 0$$

Thus we'll have $\frac{3N}{2}$ complex fermions, each of which generate a two dimensional space, so the complete space has a dimension $2^{\frac{3N}{2}}$, while the Hilbert space for N spins is 2^N-dimensional. The representation (1.33) generate replica of physical states. Below we prove that this is related with the $\mathbb{Z}_2$ symmetry of the representation. Indeed the representation (1.33) is invariant with respect to a local $\mathbb{Z}_2$ transformation:

$$\eta_i^a \to (-1)^{q_i}\, \eta_i^a \qquad \text{with } \; q_i = \pm 1 \tag{1.35}$$

The replica of states due to the enlargement of Hilbert space is absorbed in the case of $\mathbb{Z}_2$ symmetry breaking, as now we will show.
The Fourier transform of Majorana fermions is:

$$\boldsymbol{\eta_k} = \frac{1}{\sqrt{2N}} \sum_{i=1}^{N} e^{i\boldsymbol{k}\cdot\boldsymbol{R}_i} \boldsymbol{\eta}_i$$

From the reality of Majorana fermions we have:

$$\eta_{\boldsymbol{k}}^{a\,\dagger} = \eta_{-\boldsymbol{k}}^a$$

Thus in the momentum space the situation is equivalent to have $N/2$ Dirac fermions, lying in half the Brillouin zone. We can thus write:

$$\boldsymbol{\eta}_j = \sqrt{\frac{2}{N}} \sum_{k \in \frac{1}{2}BZ} \left(\boldsymbol{\eta_k} e^{i\boldsymbol{k}\cdot\boldsymbol{R}_j} + \boldsymbol{\eta}_{\boldsymbol{k}}^\dagger e^{-i\boldsymbol{k}\cdot\boldsymbol{R}_j} \right)$$

We can now notice that for each value of $\boldsymbol{k}$ two annihilating operators can be choosen, i.e. $\boldsymbol{\eta_k}$ and $\boldsymbol{\eta_{-k}}$. Therefore, in the case of symmetry breaking, there are $2^{N/2}$ equivalent way of choosing the vacuum around which developing the fluctuations. Selecting a particular one, the degeneracy of the states will disappear.

This representation can be obtained by representation (1.1) of Schwinger-Wigner, if we define another Schwinger-Wigner representation of "isospin":

$$\tau^a = \widetilde{b}^\dagger_\alpha \left(\frac{\sigma^a_{\alpha\beta}}{2}\right) \widetilde{b}_\beta \tag{1.36}$$

where $\widetilde{b}$ is the Nambu spinor:

$$\widetilde{b} = \begin{pmatrix} b_1 \\ b_2^\dagger \end{pmatrix}$$

Also this representation satisfy the $SU(2)$ algebra of spins:

$$\left[\tau^a, \tau^b\right] = i\varepsilon^{abc}\tau^c$$

but in this case the analogue of condition (1.2) to have $\boldsymbol{\tau}^2 = 3/4$ is different, indeed:

$$\boldsymbol{\tau}^2 = \frac{3}{4}\left(1 - (n_1 - n_2)^2\right) \tag{1.37}$$

so we have to impose the condition:

$$b_1^\dagger b_1 + b_2^\dagger b_2 = \begin{cases} 0 \\ 2 \end{cases} \tag{1.38}$$

Thus spin and isospin are independent and they act on different subspaces and it is easy to prove that the product of a spin operator with a isospin operator is always zero, i.e. if one is nonzero the other is automatically zero. We can define the projection operators of the two representations:

$$P^{\boldsymbol{S}} = (n_1 - n_2)^2 \qquad P^{\boldsymbol{\tau}} = 1 - (n_1 - n_2)^2$$
$$P^{\boldsymbol{S}} + P^{\boldsymbol{\tau}} = 1$$

Now we can consider the sum of the two kinds of spin and call it the spin S', with $\boldsymbol{S'} = \boldsymbol{S} + \boldsymbol{\tau}$. This continue to satisfy the commutation relations for spin, but in addition this also automatically satisfy the condition $\boldsymbol{S'}^2 = 3/4$, as we can verify:

$$\boldsymbol{S'}^2 = \boldsymbol{S}^2 + \boldsymbol{\tau}^2 + S^\alpha\tau^\alpha + \tau^\alpha S^\alpha =$$

$$= \frac{3}{4}(n_1 - n_2)^2 + \frac{3}{4}\left(1 - (n_1 - n_2)^2\right) = \frac{3}{4}$$

If we have a system of N spins we can represent each of them with either one of the two representation S and τ or with the representation S'. Using the latter we have two fermions representing each spin operator without constriction, so we have a redundancy of physical states equal to 2^N. This redundancy can be reduced introducing Majorana fermions.

We decompose the spinor b in his real and imaginary components in this way:

$$b_j = \frac{1}{2}\left(\eta^0 + i\boldsymbol{\sigma}\cdot\boldsymbol{\eta}\right)\begin{pmatrix}0\\ i\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-\eta^1 + i\eta^2\\ \eta^3 + i\eta^0\end{pmatrix}$$

where η^i are four Majorana fermions.

Now we can write explicitly spin and isospin operators in terms of complex and real fermions:

$$\begin{aligned}
S^x_j &= \frac{1}{2}\left(b^\dagger_{1,j}b_{2,j} + b^\dagger_{2,j}b_{1,j}\right) = \frac{i}{4}\left(\eta^3_j\eta^2_j + \eta^0_j\eta^1_j\right)\\
S^y_j &= \frac{i}{2}\left(b^\dagger_{2,j}b_{1,j} - b^\dagger_{1,j}b_{2,j}\right) = \frac{i}{4}\left(\eta^1_j\eta^3_j + \eta^0_j\eta^2_j\right)\\
S^z_j &= \frac{1}{2}\left(b^\dagger_{1,j}b_{1,j} - b^\dagger_{2,j}b_{2,j}\right) = \frac{i}{4}\left(\eta^2_j\eta^1_j + \eta^0_j\eta^3_j\right)\\
\tau^x_j &= \frac{1}{2}\left(b^\dagger_{1,j}b^\dagger_{2,j} + b_{2,j}b_{1,j}\right) = \frac{i}{4}\left(\eta^3_j\eta^2_j - \eta^0_j\eta^1_j\right)\\
\tau^y_j &= \frac{i}{2}\left(b_{2,j}b_{1,j} - b^\dagger_{1,j}b^\dagger_{2,j}\right) = \frac{i}{4}\left(\eta^1_j\eta^3_j - \eta^0_j\eta^2_j\right)\\
\tau^z_j &= \frac{1}{2}\left(b^\dagger_{1,j}b_{1,j} - b_{2,j}b^\dagger_{2,j}\right) = \frac{i}{4}\left(\eta^2_j\eta^1_j - \eta^0_j\eta^3_j\right)
\end{aligned} \tag{1.39}$$

If we sum spin and isospin to form the spin S', the dependence on the Majorana operator η^0 disappear, and we obtain the representation (1.33). The N fermions η^0 (one for each site) are not necessary and can be traced out, decreasing the space by $2^{N/2}$, in fact they can be recombined to form $N/2$ complex fermions. So the redundancy of physical states now is $2^N/2^{N/2} = 2^{N/2}$, as stated above.

There is also an equivalent representation of (1.33) that gives another interpretation of the replica of states. We introduce in the spin lattice a Majorana operator ψ_j for each site, which is independent from spins, so we

can write:

$$\{\psi_j, \psi_k\} = 2\delta_{jk} \, , \, \psi_j^2 = 1 \, , \quad [\psi_j, \sigma_k] = 0 \tag{1.40}$$

We can reproduce the representation (1.33) defining:

$$\eta_j^a = \sigma_j^a \psi_j \tag{1.41}$$

Now we verify that spins are correctly represented:

$$\begin{aligned} S_j^1 &= -\frac{i}{2}\eta_j^2\eta_j^3 = -\frac{i}{2}\sigma_j^2\sigma_j^3 = \frac{\sigma_j^1}{2} = S_j^1 \\ S_j^2 &= -\frac{i}{2}\eta_j^3\eta_j^1 = -\frac{i}{2}\sigma_j^3\sigma_j^1 = \frac{\sigma_j^2}{2} = S_j^2 \\ S_j^3 &= -\frac{i}{2}\eta_j^1\eta_j^2 = -\frac{i}{2}\sigma_j^1\sigma_j^2 = \frac{\sigma_j^3}{2} = S_j^3 \end{aligned}$$

and that anticommutation relations are preserved:

$$\begin{aligned} j \neq k \, , \quad & \left\{\eta_j^a, \eta_k^b\right\} = \left[\sigma_j^a, \sigma_k^b\right] \psi_j\psi_k = 0 \\ j = k \, , \quad & \left\{\eta_j^a, \eta_j^b\right\} = \left\{\sigma_j^a, \sigma_j^b\right\} \psi_j\psi_j = 2\delta^{ab} \end{aligned}$$

With the introduction of a Majorana operator the dimensionality of the Hilbert space is increased by $2^{N/2}$, this cause the Majorana representation (1.41) to have a dimension of $2^{3N/2}$.

1.3.2 Another Majorana representation

The equations (1.39) suggests another important representation of spin 1/2 in terms of four Majorana fermions.

This fermionization was firstly proposed by A. Kitaev [15] and now we propose a method to get this from the Schwinger-Wigner representation.

Four real fermions can form two complex fermions, this representation is indeed equivalent to the Schwinger-Wigner, but in some cases can be more useful.

Obviously there will be unphysical states, indeed the dimension of the Fock space exceed by 2^N. We will see that the restriction to the physical space is performed imposing a gauge invariance, as happen in electrodynamics.

Consider the new "spin" $\boldsymbol{S'} = \boldsymbol{\tau} - \boldsymbol{S}$, where τ and S are the isospin and spin operator defined before. From (1.39) S' can be written directly in terms of Majorana fermions:

$$S'^x_j = \frac{i}{2}\eta^1_j\eta^0_j \qquad S'^y_j = \frac{i}{2}\eta^2_j\eta^0_j$$
$$S'^z_j = \frac{i}{2}\eta^3_j\eta^0_j \tag{1.42}$$

Now we will explain why we called this *"spin"* and not *spin*, indeed we now proceed to verify if it satisfies the properties of spin. If we calculate S'^2, it gets the right value:

$$S'^2 = S^2 + \tau^2 - S^a\tau^a - \tau^a S^a = \frac{3}{4}$$

The commutators between spins in different sites are 0, but the algebra of the spin in the same site is not satisfied, indeed:

$$\left[S'^x_j, S'^y_j\right] = \left[\tau^x_j, \tau^y_j\right] + \left[S^x_j, S^y_j\right] = i\left(\tau^z_j + S^z_j\right) = iS'^z_j + 2iS^z_j \tag{1.43a}$$
$$\left[S'^y_j, S'^z_j\right] = iS'^x_j + 2iS^x_j \tag{1.43b}$$
$$\left[S'^z_j, S'^x_j\right] = iS'^y_j + 2iS^y_j \tag{1.43c}$$

So the conditions to be satisfied for the Majorana operators are the following:

$$S^x_j = 0 \implies \eta^3_j\eta^2_j = \eta^1_j\eta^0_j \tag{1.44a}$$
$$S^y_j = 0 \implies \eta^1_j\eta^3_j = \eta^2_j\eta^0_j \tag{1.44b}$$
$$S^z_j = 0 \implies \eta^2_j\eta^1_j = \eta^0_j\eta^3_j \tag{1.44c}$$

It easy to see that these conditions are the same one, in fact, multiplying the first for $\eta^0_j\eta^1_j$, we have:

$$\eta^0_j\eta^1_j\eta^3_j\eta^2_j = 1 \implies \eta^1_j\eta^2_j\eta^3_j\eta^0_j = 1$$

Doing the same with the other two, we always obtain the relation above. So, if we call $D_j = \eta^1_j\eta^2_j\eta^3_j\eta^0_j$, we have that applying it to physical states, it

must give unity in order to maintain the algebra of spins:

$$D_j \left|\Psi\right\rangle_{\text{phys}} = \left|\Psi\right\rangle_{\text{phys}} \tag{1.45}$$

This relation is equivalent to the Gauss law for electrodynamics. The physical space is the gauge invariant sector.

From the conditions (1.44) we can see that this representation is indeed equivalent to the isospin, because the spin is zero. In particular the gauss law (1.45) that we must impose is equivalent to the constraint (1.38), i.e. the two possible states of spin correspond to a fully occupied fermion state or an empty one.

1.3.3 Application to the Heisenberg model for two spin 1/2

Consider two spin interacting with Heisenberg coupling (taking the exchange constant to be 1):

$$H = \boldsymbol{S}_1 \cdot \boldsymbol{S}_2 \tag{1.46}$$

We know that the spectrum has two level, corresponding to the singlet and the triplet states of the spins. The ground state is the singlet state and it has no degeneracy, while the triplet state has a three-fold degeneracy. Using the standard Majorana representation expressed in (1.33), we rewrite the Hamiltonian:

$$\begin{aligned} H &= -\frac{1}{16}\varepsilon_{abc}\varepsilon_{ade}\eta_1^b\eta_1^c\eta_2^d\eta_2^e = -\frac{1}{16}\left(\delta_{bd}\delta_{ce} - \delta_{be}\delta_{cd}\right)\eta_1^b\eta_1^c\eta_2^d\eta_2^e = \\ &= -\frac{1}{16}\Big(-\eta_1^b\eta_2^b\eta_1^c\eta_2^c - \underbrace{\eta_1^b\eta_1^c}_{2\delta_{bc}-\eta_1^c\eta_1^b}\eta_2^c\eta_2^b\Big) = -\frac{1}{16}\left(-2\left(\boldsymbol{\eta}_1\cdot\boldsymbol{\eta}_2\right)^2 - 6\right) = \\ &= \frac{1}{8}\left(3 + \left(\boldsymbol{\eta}_1\cdot\boldsymbol{\eta}_2\right)^2\right) \end{aligned} \tag{1.47}$$

So we have six Majorana fermions that can be combined to form three complex fermions on the bond:

$$\begin{aligned} \boldsymbol{f}^\dagger &\equiv \frac{1}{2}\left(\boldsymbol{\eta}_1 + i\boldsymbol{\eta}_2\right) \\ \boldsymbol{f} &\equiv \frac{1}{2}\left(\boldsymbol{\eta}_1 - i\boldsymbol{\eta}_2\right) \end{aligned} \tag{1.48}$$

These fermions act on a Hilbert space of dimension $2^3 = 8$, so the dimensionality of the original spin space has increased by a factor 2.

The 'unphysical' states are those that are generated by the Majorana operators ψ_j introduced in (1.41).

In the representation above 'physical' and 'unphysical' states are mixed, but, as we will see, since the original Hamiltonian acts only on physical states, 'unphysical' one factor out and increase the degeneracy of the spectrum.

The Hamiltonian (1.46) in terms of the fermions $\boldsymbol{f}$ becomes:

$$H = \frac{3}{8} - \frac{1}{2}\left(\boldsymbol{f}^\dagger \cdot \boldsymbol{f} - \frac{3}{2}\right)^2 \tag{1.49}$$

Thus it is easy to get the spectrum with his degeneracy. We write the possible eigenvalues for $\boldsymbol{f}^\dagger \cdot \boldsymbol{f}$, with the energy and the degeneracy:

$\boldsymbol{f}^\dagger \cdot \boldsymbol{f}$	E	Degeneracy
0	$-\frac{3}{4}$	1
1	$\frac{1}{4}$	3
2	$\frac{1}{4}$	3
3	$-\frac{3}{4}$	1

So the spectrum is given by:

$$\begin{aligned} E_0 &= -\frac{3}{4}\,, \quad \text{2-fold degeneracy} \\ E_1 &= \frac{1}{4} \quad\;, \quad \text{6-fold degeneracy} \end{aligned} \tag{1.50}$$

corresponding to the singlet and the triplet state, but with a double degeneracy, as expected.

This is related also to the invariance of the transformed Hamiltonian with respect to $\mathbb{Z}_2$ local transformations $\boldsymbol{\eta}_i \to -\boldsymbol{\eta}_i$, which is reflected in the particle-hole symmetry $\boldsymbol{f}^\dagger \to \boldsymbol{f}$ for the complex fermions.

As an example we now develop a mean field treatment of this simple model. We define the following average on the link:

$$V \equiv \left\langle \boldsymbol{f}^\dagger \cdot \boldsymbol{f} - \frac{3}{2} \right\rangle = -i \left\langle \boldsymbol{\eta}_1 \cdot \boldsymbol{\eta}_2 \right\rangle \tag{1.51}$$

and the fluctuation around it:

$$\delta \equiv : \boldsymbol{f}^\dagger \cdot \boldsymbol{f} - \frac{3}{2} : = \left(\boldsymbol{f}^\dagger \cdot \boldsymbol{f} - \frac{3}{2} \right) - V \tag{1.52}$$

Neglecting fluctuations of second order the Hamiltonian becomes:

$$H = \frac{3}{8} - \frac{V^2}{2} - V\delta = \frac{3}{8} - V \left(\boldsymbol{f}^\dagger \cdot \boldsymbol{f} - \frac{3}{2} \right) + \frac{V^2}{2} \tag{1.53}$$

The aforementioned $\mathbb{Z}_2$ symmetry is broken. We can find the ground state and then check for consistency the value of V.

There are two possible case, $V > 0$ or $V < 0$. If $V > 0$ then the ground state correspond to the state with all the fermions filled, i.e. $\boldsymbol{f}^\dagger \cdot \boldsymbol{f} \, |g\rangle = 3 \, |g\rangle$. Calculating V from this we obtain, $V = 3 - 3/2 = 3/2$, so the energy of the ground state is:

$$E_0 = \langle H \rangle = -\frac{3}{4} \, .$$

Thus we have obtained not only the exact value for the ground state energy, but also the right degeneracy.
In fact the state with all the fermions filled is one-fold degenerate, as is the singlet state. In general, for an antiferromagnetic spin chain, the mean field treatment gives only an approximation for the energy, but gives the correct ground state degeneracy.

This result is in agreement with the discussion made above, about the replica of the states.

Chapter 2

Kitaev model on a honeycomb lattice

In this chapter we apply the formalism of fermionization to a recent model, introduced for the first time by A. Kitaev [15]. This is an example of a system that exhibits topological behavior.

We will now describe the model and its symmetries, obtaining an important conserved quantity on plaquettes, which permits the exact solution of the model.

In the following sections we will apply two kind of fermionization to diagonalize the Hamiltonian.

In particular first we will solve it using the method proposed by Kitaev, using Majorana fermions, then we will apply a Jordan-Wigner transformation.

These two methods give obviously the same results, but the former generate unphysical states, while the latter gives exact solutions. Thus if we are interested on wave functions the Jordan-Wigner representation is more useful.

But if we want, for example, to study unpaired Majorana modes on the boundary it is convenient to use the second one.

2.1 The model

This model is defined on a 2-dimensional honeycomb lattice. It is very special because of his topology.

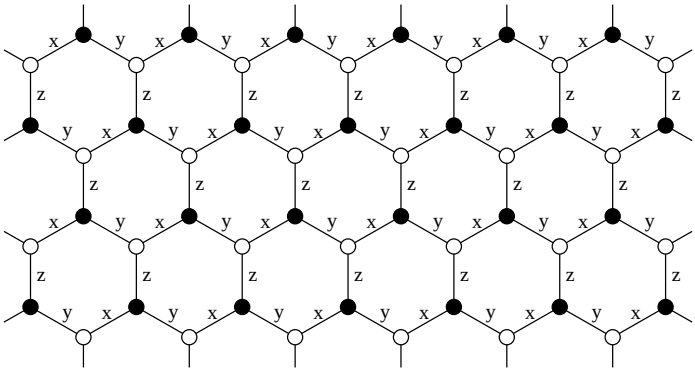


Figure 2.1: The honeycomb lattice with three types of links and the black and white sub-lattices

The hexagonal lattice can be viewed as two overlapping triangular sub-lattices in which the vertexes of one stay in the plaquettes' center of the other. We will denote one as the white (w) sub-lattice and the other as the black (b) sub-lattice, refer to fig. 2.1.

In this model there are only interactions between nearest neighbors and there are three types of links (x, y, z), depending on the direction of link.

The Kitaev Hamiltonian is (see fig. 2.1):

$$H = -J_x \sum_{x\text{-bonds}} \sigma^x_{R_w}\sigma^x_{R_b} - J_y \sum_{y\text{-bonds}} \sigma^y_{R_w}\sigma^y_{R_b} - J_z \sum_{z\text{-bonds}} \sigma^z_{R_w}\sigma^z_{R_b} \qquad (2.1)$$

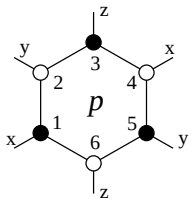
where R_w (R_b) denote the site in the white (black) sub-lattice and $\sigma^{x,y,z}$ are the Pauli matrix. The sum are over the links, so each nearest neighbor interaction is taken once. It can be noted that x-bonds always goes from white to black (left-right), y-bonds goes from black to white, while z-bonds goes from white to black (up-down).

Furthermore the model renders a high degree of frustration, as we can understand even at a classical level, indeed a given spin cannot satisfy conflicting demands of orientation from 3 nearest neighbors.

2.1.1 An integral of motion

In order to diagonalize this Hamiltonian we now investigate for symmetries. Consider a single plaquette p of figure below and the product of links

around it:



$$I_p = \sigma^z_{1_b}\sigma^z_{2_w}\sigma^x_{2_w}\sigma^x_{3_b}\sigma^y_{3_b}\sigma^y_{4_w}\sigma^z_{4_w}\sigma^z_{5_b}\sigma^x_{5_b}\sigma^x_{6_w}\sigma^y_{6_w}\sigma^y_{1_b} \tag{2.2}$$

that, using the relation $\sigma^a\sigma^b = i\varepsilon^{abc}\sigma^c$, reduces to:

$$I_p = \sigma^x_{1_b}\sigma^y_{2_w}\sigma^z_{3_b}\sigma^x_{4_w}\sigma^y_{5_b}\sigma^z_{6_w} \tag{2.3}$$

i.e. the product of the "external legs" of the hexagon.

Now we show that these quantities defined on each plaquette are conserved. First they commutes with the link terms. For a bond on the plaquette:

$$\left[I_p, \sigma^z_{1_b}\sigma^z_{2_w}\right] = \sigma^y_{1_b}\sigma^x_{2_w}\sigma^z_{3_b}\sigma^x_{4_w}\sigma^y_{5_b}\sigma^z_{6_w} - \sigma^y_{1_b}\sigma^x_{2_w}\sigma^z_{3_b}\sigma^x_{4_w}\sigma^y_{5_b}\sigma^z_{6_w} = 0$$

while for a bond external to the plaquette, i.e. a "leg":

$$\left[I_p, \sigma^z_{3_b}\sigma^z_{7_w}\right] = \sigma^x_{1_b}\sigma^y_{2_w}\sigma^z_{7_w}\sigma^x_{4_w}\sigma^y_{5_b}\sigma^z_{6_w} - \sigma^x_{1_b}\sigma^y_{2_w}\sigma^z_{7_w}\sigma^x_{4_w}\sigma^y_{5_b}\sigma^z_{6_w} = 0.$$

The same thing happens for other bonds, so the I_p's commutes with each other, from (2.2), and with the Hamiltonian, thus they are all conserved quantities.

Eigenvalues for these operators can be found by noting that

$$I_p^2 = 1$$

so the eigenvalues are

$$\mathcal{I}_p = \pm 1. \tag{2.4}$$

For reasons that will become clear later we say that if $\mathcal{I}_p = -1$ then there is a vortex in the plaquette p, else if $\mathcal{I}_p = +1$ there isn't any vortex.

As we said above, I_p's are conserved, so we can solve the Hamiltonian for each vortex configuration, i.e. a set of eigenvalues $\mathcal{I}_p$ defined on each plaquette. The variables $\mathcal{I}_p$ are nothing else than a static $\mathbb{Z}_2$ Ising field.

Furthermore we can also note another property of the model, indeed, for

a compact surface, e.g. if the system is on a torus, it must be

$$\prod_p I_p = 1 \tag{2.5}$$

in fact if we consider a z-bond $1_b - 2_w$ and we multiply the four plaquette operators adjacent to it in the order left-right and up-down we have for the site 1_b and 2_w:

$$\sigma^z_{2_w}\sigma^x_{2_w}\sigma^y_{2_w}\sigma^y_{1_b}\sigma^x_{1_b}\sigma^z_{1_b} = (-i)i = 1$$

This can be done for all the z-bonds up to cover all the lattice, obtaining (2.5). From this result we can understand that vortex will come in pair, in order to keep the (2.5) true.

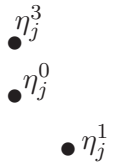
To easily diagonalize this system we proceed with fermionization, but we will see that this procedure also bring to other results that we will emphasize.

2.2 Fermionization using Majorana fermions

This is the fermionization that was proposed in the work of Kitaev [18].

We make use of the fermionization method described in §1.3.2, that introduces four Majorana fermions η^a for each site, see (1.42), and a constraint, expressed as a gauge invariance on the physical subspace, see (1.45).

To visually see the fermionization, we will use this particular graphical representation, splitting each site in four:



$$\sigma^x_j = i\eta^1_j\eta^0_j\,, \quad \sigma^y_j = i\eta^2_j\eta^0_j\,, \quad \sigma^z_j = i\eta^3_j\eta^0_j \tag{2.6}$$

For simplicity we rename the Majorana operators in this way, $\eta^1 \to b^x$, $\eta^2 \to b^y$, $\eta^3 \to b^z$ and $\eta^0 \to c$, so that the transformation correspond to:

$$\sigma^x_j = ib^x_j c_j\,, \quad \sigma^y_j = ib^y c_j\,, \quad \sigma^z_j = ib^z_j c_j$$

The spin operators are now expressed in the extended Hilbert space introduced by the fermionization. Projection into physical states is made through the operator D_j.

As we will see below, $[D_j, \sigma^a_k] = 0$ in the extended Hilbert space, and as

a consequence it commutes with the Hamiltonian too.

So the spectrum of the Hamiltonian (2.1) in the extended space is the same, and in particular each vortex configuration has the same spectrum. The transformation introduces gauge copies of physical states.

We now prove that spins in the extended space are gauge invariant, i.e. $[D_j, \sigma_k^a] = 0$. For $j \neq k$ it is trivial because each operator is composed by an even number of different fermions, so they commute. While for $j = k$:

$$\begin{aligned}\left[D_j, \sigma_j^x\right] &= i(b_j^x b_j^y b_j^z c_j b_j^x c_j - b_j^x c_j b_j^x b_j^y b_j^z c_j) = \\ &= i(-b_j^x b_j^y + b_j^x b_j^y) = 0\end{aligned}$$

and the same happens for $\sigma_j^{(y,z)}$.

So we can get the spectrum diagonalizing the fermionized Hamiltonian. Transforming the exchange terms, we have:

$$x\text{-bonds} \qquad \sigma_j^x \sigma_k^x = (i)^2 b_j^x c_j b_k^x c_k = -i(i b_j^x b_k^x) c_j c_k \tag{2.7a}$$

$$y\text{-bonds} \qquad \sigma_j^y \sigma_k^y = (i)^2 b_j^y c_j b_k^y c_k = -i(i b_j^y b_k^y) c_j c_k \tag{2.7b}$$

$$z\text{-bonds} \qquad \sigma_j^z \sigma_k^z = (i)^2 b_j^z c_j b_k^z c_k = -i(i b_j^z b_k^z) c_j c_k \tag{2.7c}$$

Where we have enclosed between parenthesis an operator, defined on the bonds:

$$\hat{u}_{\langle ij \rangle_a} = i b_i^a b_j^a \tag{2.8}$$

where i and j correspond to neighboring sites connected by a-link.
We can write the transformed Hamiltonian:

$$H = i \sum_{\langle j,k \rangle} J_{\alpha_{\langle j,k \rangle}} \hat{u}_{\langle j,k \rangle} c_j c_k \tag{2.9}$$

where $\alpha_{\langle i,j \rangle} = (x, y, z)$ depending on the type of link and the sum is taken on the links.

We now proceed to verify what does the $\hat{u}_{\langle i,j \rangle}$ represent. It is easy to verify some properties:

$$\hat{u}_{\langle i,j \rangle}^\dagger = \hat{u}_{\langle i,j \rangle} \qquad \hat{u}_{\langle i,j \rangle}^2 = 1 \tag{2.10}$$

and that they commutes each other and with the Hamiltonian:

$$\left[\hat{u}_{\langle i,j\rangle}, \hat{H}\right] = 0\,, \qquad \left[\hat{u}_{\langle i,j\rangle}, \hat{u}_{\langle k,l\rangle}\right] = 0 \tag{2.11}$$

because a couple of fermions always commute with a couple of different fermions, and in fact the Hamiltonian contains only couple of different fermions, apart the operator itself.

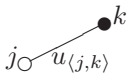
This is a feature of this model and of this fermionization, and is the key feature that make this model so important. There is a sort of recombination of Majorana operators on the links, that simplify a lot the model, but at the same time give important features at the boundary, where can remain unpaired fermions.

From the properties above we can make the following statements: the $\hat{u}_{\langle i,j\rangle}$ are conserved quantities and the possible eigenvalues are $u_{\langle i,j\rangle} = \pm 1$. Thus one can resolve the spectrum of the Hamiltonian for each configuration of $u_{\langle i,j\rangle}$ and the extended Hilbert space ($\mathcal{L}$) splits into subspaces $\mathcal{L}_u$, indexed by the configuration of $u_{\langle j,k\rangle}$. This results in the following Hamiltonian:

$$H = \frac{1}{4}\sum_{j,k} A_{jk} c_j c_k \tag{2.12}$$

that is the Hamiltonian of free Majorana fermions in a static $\mathbb{Z}_2$ background field, where we have extended the sum to all sites introducing the c-number matrix $A_{jk} = i2u_{\langle j,k\rangle}J_{\alpha_{\langle j,k\rangle}}$, if j and k are connected by a link and $A_{jk} = 0$ otherwise. The further factor $1/2$ is introduced to avoid replicated terms.

It is important to note that the subspaces $\mathcal{L}_u$ are not gauge invariant, indeed the Gauge transformation D_j change the sign to all the $u_{\langle j,k\rangle}$ connected to the site j, for example, take a x-link:



$$D_j \hat{u}_{\langle j,k\rangle} D_j = Ibo_j^x b_j^y b_j^z c_j b_j^x b_k^x D_j = -ib_j^x b_k^x D_j D_j = -\hat{u}_{\langle j,k\rangle} \tag{2.13}$$

so for practical purpose we will say that the local gauge transformation acting on the fields u gives $u_{\langle j,k\rangle} \to \tau_j u_{\langle j,k\rangle}\tau_k$, where $\tau_j = \pm 1$ depending whether the gauge transformation D_j is acting or not.

Since the Hamiltonian is invariant under the gauge transformation, gauge equivalent subspaces have the same spectrum. How are they connected to

the physical subspace, i.e. the gauge invariant sector?

In section 2.1 we decomposed the physical Hilbert space into subspaces, labeled by vortex configurations. In the extended Hilbert space the vortex operators commutes with the gauge transformations D_j, since the latter commutes with spin operators, so the *vortex configuration is left unchanged by gauge transformations.*

We take the following vortex operators in the extended Hilbert space, for a specific plaquette p:

$$\hat{W}_p = \prod_{\langle j,k\rangle \in \partial p} \hat{u}_{\langle j,k\rangle} \qquad \begin{array}{l} j \in \text{white sub-lattice} \\ k \in \text{black sub-lattice} \end{array} \tag{2.14}$$

where ∂p indicates the boundary of the plaquette p. We now verify that that W_p reduces to I_p, defined in (2.3), when restricted to the physical space. Using the same labeling of (2.2):

$$\begin{aligned} \hat{W}_p &= \hat{u}_{\langle 2,1\rangle}\hat{u}_{\langle 2,3\rangle}\hat{u}_{\langle 4,3\rangle}\hat{u}_{\langle 4,5\rangle}\hat{u}_{\langle 6,5\rangle}\hat{u}_{\langle 6,1\rangle} = \\ &= -(i)^6 b_1^z\ b_2^z\ b_2^x\ b_3^x\ b_3^y\ b_4^y\ b_4^z\ b_5^z\ b_5^x\ b_6^x\ b_6^y\ b_1^y = \end{aligned}$$

using $ib_j^z b_j^x = -\sigma_j^y D_j$ and cyclic permutations,

$$= \sigma_1^x D_1 \sigma_2^y D_2 \sigma_3^z D_3 \sigma_4^x D_4 \sigma_5^y D_5 \sigma_6^z D_6$$

This is equal to I_p in the physical subspace, where all $D_j = 1$.

The relation (2.14) in terms of the eigenvalues is:

$$w_p = \prod_{\langle j,k\rangle \in \partial p} u_{\langle j,k\rangle} \tag{2.15}$$

thus, taking $w_p = 1$ it is gauge equivalent to have no-vortex in the physical plaquette p, while if $w_p = -1$ it is gauge equivalent to have a vortex.

We are now able to solve the spectrum for each single vortex configuration by choosing any configuration of $u_{\langle j,k\rangle}$, that give the correct vortex configuration, this operation is the same as fixing a gauge.

Before going ahead to the calculation of the spectrum, we make some clarifications. Although this approach is very useful for finding the energy levels, it is not so simple to get the eigenstates of the Hamiltonian. Indeed,

since the subspace $\mathcal{L}_u$ is not preserved by the gauge transformations, the eigenstates are not preserved too, so they cannot be physical states, since from (1.45) they must be gauge invariant. In order to get the physical state we must symmetrize over all gauge transformations:

$$|\Psi\rangle_{\text{phys}} = \prod_j \left(\frac{1+D_j}{2}\right)|\Psi\rangle \tag{2.16}$$

where $|\Psi\rangle$ is an eigenstate of the Hamiltonian (2.12).

2.2.1 Spectrum of the vortex-free state

In this section we will perform an explicit calculation of the spectrum of (2.12) for a particularly simple configuration, that with no vortex, i.e. with all $w_p = 1$.

This configuration is the one with lowest energy ground state, as showed by a beautiful theorem by E. Lieb [28], so it is the true ground state of the model.

From now on we will consider $J_x, J_y, J_z \geqslant 0$. It is easy to see that the ground state does not depend on the sign of exchange constants, indeed given a configurations of $u_{\langle j,k\rangle}$, changing the sign of one of the constants, e.g. J_z, it is equivalent to change the sign of $u_{\langle j,k\rangle}$ for all the z-links. But this new configuration is gauge equivalent to the precedent, so the spectrum is the same.

In order to diagonalize the Hamiltonian, we must fix the gauge. The simplest choice is:

$$u_{\langle j,k\rangle} = 1 \quad \text{for all } \langle j,k\rangle \text{ with } \begin{array}{l} j \in \text{white sub-lattice} \\ k \in \text{black sub-lattice} \end{array} \tag{2.17}$$

so that $w_p = 1$ for all p.

It is now convenient to redefine a unit cell and a new lattice, so as to make the Hamiltonian traslational invariant.
This is done by considering a z-link as the new unit cell and it will be indexed by s or t. Each one of the two sites in the unit cell will be identified by a index λ or $\mu = 1, 2$, where 1 stand for the white site and 2 for the black one, see fig. 2.2.

In this way the old index j that identifies the site becomes:

$$j \longrightarrow (s, \lambda) \tag{2.18}$$

The Hamiltonian can be written:

$$H = \frac{1}{4} \sum_{\substack{s,t \\ \lambda,\mu}} A_{s\lambda,t\mu} c_{s\lambda} c_{t\mu} \tag{2.19}$$

For simplicity this new lattice can be mapped to a square lattice, of unit vectors $\boldsymbol{n}_x, \boldsymbol{n}_y$.

This configuration is translational invariant along both x and y directions, for the particularly set of $u_{\langle j,k \rangle}$, and can be solved by Fourier transforming. Let us define the Fourier transform of Majorana fermions:

$$c_\lambda(\boldsymbol{q}) = \frac{1}{\sqrt{N}} \sum_s e^{i\boldsymbol{q}\cdot\boldsymbol{r}_s} c_{s\lambda} \tag{2.20}$$

$$c_\lambda^\dagger(\boldsymbol{q}) = \frac{1}{\sqrt{N}} \sum_s e^{-i\boldsymbol{q}\cdot\boldsymbol{r}_s} c_{s\lambda}^\dagger \tag{2.21}$$

where N is the number of cells, so that the total number of sites is $2N$, and the momentum q belong to a torus with $q_{x,y} \in [-\pi, \pi]$. The fermion operators in momentum representation satisfies the following algebra:

$$\left\{ c_\lambda(\boldsymbol{q}), c_\mu^\dagger(\boldsymbol{p}) \right\} = 2\delta_{\lambda\mu}\delta_{\boldsymbol{qp}} \tag{2.22}$$

Furthermore, recalling the reality condition of Majorana fermions $c_{s\lambda}^\dagger = c_{s\lambda}$, we have that $c_\lambda^\dagger(\boldsymbol{q}) = c_\lambda(-\boldsymbol{q})$.

Let us evaluate the matrix $A_{\lambda\mu}(s,t) \equiv A_{r\lambda,s\mu}$. The elements A_{11} and A_{22} are identically 0 for all s, t because there is never any hopping between sites of the same color. While non diagonal terms are different from zero if $s = t$ or if s, t are nearest neighbors, but they don't depend on the sites, i.e. they are traslationally invariant.
So we can define the Fourier transform for each non diagonal term, referring

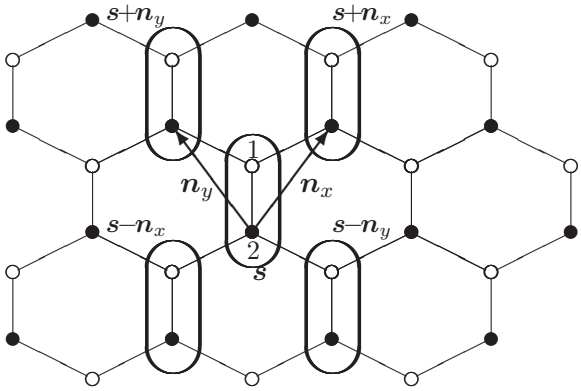


Figure 2.2: The new unit cells connected by the new lattice vectors $\boldsymbol{n}_x$, $\boldsymbol{n}_y$

to fig 2.2:

$$A_{12}(\boldsymbol{q}) = \sum_t e^{i\boldsymbol{q}\cdot(\boldsymbol{r_t}-\boldsymbol{r_s})} A_{s1,t2} = i\left(2e^{iq_x}J_x + 2e^{iq_y}J_y + 2J_z\right) \tag{2.23}$$

$$A_{21}(\boldsymbol{q}) = \sum_t e^{i\boldsymbol{q}\cdot(\boldsymbol{r_t}-\boldsymbol{r_s})} A_{s2,t1} = -i\left(2e^{-iq_x}J_x + 2e^{-iq_y}J_y + 2J_z\right) \tag{2.24}$$

where we have used that $u_{\langle s1,t2\rangle} = 1$ for links starting from $s1$ and $u_{\langle s2,t1\rangle} = -1$ for links starting from $s2$. So we can define the Fourier transform of the whole matrix:

$$A_{\lambda\mu}(\boldsymbol{q}) = \sum_t e^{i\boldsymbol{q}\cdot(\boldsymbol{r}_t-\boldsymbol{r}_s)} A_{s\lambda,t\mu} = \left[\begin{pmatrix} 0 & if(\boldsymbol{q}) \\ -if^*(\boldsymbol{q}) & 0 \end{pmatrix}\right]_{\lambda\mu} \tag{2.25}$$

and the inverse transform:

$$A_{s\lambda,t\mu} = \frac{1}{N}\sum_{\boldsymbol{q}} e^{-i\boldsymbol{q}\cdot(\boldsymbol{r}_t-\boldsymbol{r}_s)} A_{\lambda\mu}(\boldsymbol{q}) \tag{2.26}$$

In the precedent equations we have defined:

$$f(\boldsymbol{q}) = 2\left(e^{iq_x}J_x + e^{iq_y}J_y + J_z\right) \tag{2.27}$$

which has the property:

$$f^*(\boldsymbol{q}) = f(-\boldsymbol{q}) \tag{2.28}$$

The eigenvalues of that matrix are $\pm|f(\boldsymbol{q})|$, that correspond to a filled and an empty band.

Substituting the inverse Fourier transforms into Hamiltonian (2.19) we obtain:

$$H = \frac{1}{4}\sum_{\boldsymbol{q}}\sum_{\lambda\mu} A_{\lambda\mu}(\boldsymbol{q})c^{\dagger}_{\lambda}(\boldsymbol{q})c_{\mu}(\boldsymbol{q}) = \frac{1}{4}\sum_{\boldsymbol{q}}\Big(if(\boldsymbol{q})c^{\dagger}_{1}(\boldsymbol{q})c_{2}(\boldsymbol{q}) + h.c.\Big) \quad (2.29)$$

Being $f(\boldsymbol{q})$ a complex number we can write it as $|f(\boldsymbol{q})|\, e^{i\varphi}$. In order to get the excitation spectrum we make the following transformation introducing the complex fermions $b^{\dagger}(\boldsymbol{q})$, defined on each z-bond:

$$b\ (\boldsymbol{q}) = \frac{1}{2}\left[c_1(\boldsymbol{q}) + ie^{i\varphi}c_2(\boldsymbol{q})\right] \quad (2.30a)$$

$$\begin{aligned} b^{\dagger}(\boldsymbol{q}) &= \frac{1}{2}\left[c^{\dagger}_1(\boldsymbol{q}) - ie^{-i\varphi}c^{\dagger}_2(\boldsymbol{q})\right] = \\ &= \frac{1}{2}\left[c_1(-\boldsymbol{q}) - ie^{-i\varphi}c_2(-\boldsymbol{q})\right] \end{aligned} \quad (2.30b)$$

satisfying the Dirac algebra:

$$\Big\{b(\boldsymbol{q}), b^{\dagger}(\boldsymbol{k})\Big\} = \frac{1}{4}\Big[\big\{c_1(\boldsymbol{q}), c^{\dagger}_1(\boldsymbol{k})\big\} + \big\{c_2(\boldsymbol{q}), c^{\dagger}_2(\boldsymbol{k})\big\}\Big] = \delta_{\boldsymbol{qk}}$$

from which we derive the following inverse transformations:

$$c_1(\boldsymbol{q}) = b(\boldsymbol{q}) + b^{\dagger}(-\boldsymbol{q}) \quad (2.31a)$$

$$ie^{i\varphi}c_2(\boldsymbol{q}) = b(\boldsymbol{q}) - b^{\dagger}(-\boldsymbol{q}) \quad (2.31b)$$

Substituting in (2.29) we have:

$$\begin{aligned} H = \frac{1}{4}\sum_{\boldsymbol{q}}|f(\boldsymbol{q})|\Big[&b^{\dagger}(\boldsymbol{q})b(\boldsymbol{q}) + b^{\dagger}(-\boldsymbol{q})b(-\boldsymbol{q}) - 1 + \\ &+ b^{\dagger}(\boldsymbol{q})b(\boldsymbol{q}) + b^{\dagger}(-\boldsymbol{q})b(-\boldsymbol{q}) - 1\Big] \end{aligned}$$

and using the property (2.28) we can change the sign of q without problem, obtaining:

$$H = \sum_{\boldsymbol{q}}|f(\boldsymbol{q})|\left(b^{\dagger}(\boldsymbol{q})b(\boldsymbol{q}) - \frac{1}{2}\right) \quad (2.32)$$

Thus we have that quasi-particle in this model are complex fermions with

spectrum of excitations:

$$\begin{aligned}\varepsilon(\boldsymbol{q}) &= |f(\boldsymbol{q})| = \\ &= 2\sqrt{J_x^2 + J_y^2 + J_z^2 + 2J_xJ_y\cos(q_x - q_y) + 2J_xJ_z\cos q_x + 2J_yJ_z\cos q_y}\end{aligned} \tag{2.33}$$

and a ground state:

$$\begin{aligned}|gs\rangle &= \prod_{\boldsymbol{q}} b(\boldsymbol{q})\,|0\rangle = \prod_{\boldsymbol{q}} \left(c_1(\boldsymbol{q}) + ie^{i\varphi}c_2(\boldsymbol{q})\right)|0\rangle = \\ &= \prod_{\boldsymbol{q}\in\frac{1}{2}\text{B.Z.}} \left(c_1^\dagger(\boldsymbol{q}) + ie^{i\varphi}c_2^\dagger(\boldsymbol{q})\right)|0\rangle\end{aligned} \tag{2.34}$$

where $|0\rangle$ is the vacuum of Majorana fermions in one half of the first Brillouin zone.
However it is worth remembering remember that this is not the true physical ground state because of this state is not gauge invariant.
The energy of the ground state is:

$$E_{gs} = -\frac{1}{2}\sum_{\boldsymbol{q}} |f(\boldsymbol{q})| \tag{2.35}$$

and it corresponds to a filled spectrum with negative energies.

It is now interesting to verify whether the spectrum is gapless or not, i.e. if the spectrum of excitations take the value zero for some momentum.

We must solve the equation $|f(\boldsymbol{q})| = 0$. Taking the complex value of $f(\boldsymbol{q})$, it is equivalent to solve:

$$J_x e^{iq_x} + J_y e^{iq_y} + J_z = 0 \tag{2.36}$$

that can be separate in its real and imaginary part:

$$\begin{aligned}J_x\cos q_x + J_y\cos q_y + J_z &= 0 \\ J_x\sin q_x + J_y\sin q_y &= 0\end{aligned} \tag{2.37}$$

Now we suppose that all the exchange constant are strictly > 0. Other cases will be considered later. Taking $q_x \geqslant 0$ and $q_y \leqslant 0$ or in a specular way

$q_x \leqslant 0$ and $q_y \geqslant 0$ we can get from the second equation:

$$J_x \sin q_x = J_y \sqrt{1 - \cos^2 q_y} \Longrightarrow$$
$$J_y^2 \cos^2 q_y = J_y^2 - J_x^2 \sin q_x$$

Now manipulating the first equation:

$$J_y \cos q_y = -J_z - J_x \cos q_x$$

squaring and substituting the result above we have:

$$J_y^2 - J_x^2 \sin^2 q_x = J_z^2 + J_x^2 \cos^2 q_x + 2J_x J_z \cos q_x \Longrightarrow$$
$$\cos q_x = \frac{J_y^2 - J_x^2 - J_z^2}{2J_x J_z}$$
$$\cos q_y = \frac{J_x^2 - J_y^2 - J_z^2}{2J_y J_z}$$

So the solution for q_x and q_y are:

$$q_x = \pm \arccos\left(\frac{J_y^2 - J_x^2 - J_z^2}{2J_x J_z}\right) \tag{2.38a}$$
$$q_y = \mp \arccos\left(\frac{J_x^2 - J_y^2 - J_z^2}{2J_y J_z}\right) \tag{2.38b}$$

The solution exists, but not for all the values of the exchange constants, in fact they must satisfy the following four disequations:

$$-2J_x J_z \overset{(1)}{\leqslant} J_y^2 - J_x^2 - J_z^2 \overset{(2)}{\leqslant} 2J_x J_z \tag{2.39}$$
$$-2J_y J_z \overset{(3)}{\leqslant} J_x^2 - J_y^2 - J_z^2 \overset{(4)}{\leqslant} 2J_y J_z \tag{2.40}$$

From the second we can obtain:

$$J_y^2 \leqslant (J_x + J_z)^2 \Longrightarrow J_y \leqslant J_x + J_z \tag{2.41a}$$

while from the fourth:

$$J_x^2 \leqslant (J_y + J_z)^2 \Longrightarrow J_x \leqslant J_x + J_z \tag{2.41b}$$

The last relation can be obtained summing the first and the third inequalities:

$$2J_z^2 \leqslant 2J_xJ_z + 2J_yJ_z \Longrightarrow J_z \leqslant J_x + J_y \tag{2.41c}$$

Thus the spectrum is gapless if the exchange constants satisfy the triangle inequalities (2.41).

So, when one of the inequalities becomes an equality, there is a sort of phase transition between a gapless and a gapful phase. It is indeed a quantum phase transition, because we are at zero temperature, but it isn't a traditional phase transition driven by a local order parameter, such as the magnetization for a magnet. We will return to this argument in next sections.

Now it is interesting to visualize the phase diagram. To do this we restrict to the plane $J_x + J_y + J_z = 1$, without losing of generality, because the only important quantities are the ratio between exchange constants, so there are only two independent parameter. On that plane the transition will happen along the following straight lines:

$$\begin{aligned} J_x + J_y = J_z \quad &\Longrightarrow \quad J_z = \frac{1}{2} \\ J_y + J_z = J_x \quad &\Longrightarrow \quad J_x = \frac{1}{2} \\ J_x + J_z = J_y \quad &\Longrightarrow \quad J_y = \frac{1}{2} \end{aligned}$$

In fig. 2.3 it is reproduced the phase diagram. There are interesting properties, the first is that there is an extended gapless phase, such as in the XY model, but differently there aren't long range correlations between spins, as we will see. Furthermore the three gapful phases A_x, A_y and A_z are different phase because there isn't any way to go continuously from one to the other, although they are related each other by rotational symmetry.

We now consider the gapless phase. By the solutions (2.38) we know that, if all the exchange constant are non-zero, there are two solutions for which $\varepsilon(\boldsymbol{q}) = 0$, we call them $\boldsymbol{q} = \pm\boldsymbol{q}_*$.

Consider for example the point at the center of the gapless phase,

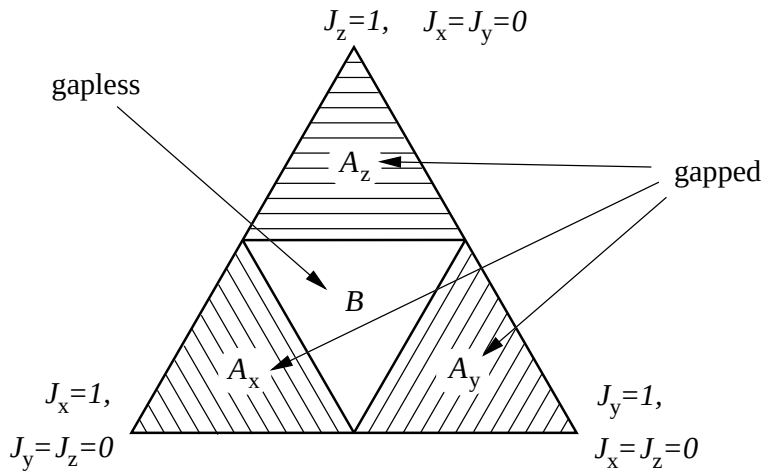


Figure 2.3: Phase diagram of the vortex-free model. The triangle is the part of the plane $J_x + J_y + J_z = 1$ in the octant $J_x > 0, J_y > 0, J_z > 0$. There are four phases, three of which are gapful (A_x,A_y,A_z). They are separated by a gapless phase B

i.e. $J_x = J_y = J_z$. The points where $\varepsilon(\boldsymbol{q}) = 0$ are:

$$\boldsymbol{q}_* = \left(\tfrac{2\pi}{3}\,, \quad -\tfrac{2\pi}{3}\right) \tag{2.42}$$

$$-\boldsymbol{q}_* = \left(-\tfrac{2\pi}{3}\,, \quad \tfrac{2\pi}{3}\right) \tag{2.43}$$

The spectrum is showed in fig 2.4.

If we move towards the phase boundary, the two points move till they fuse at the phase boundary, in fact if, for example we are on the boundary $J_x + J_z = J_y$, we have:

$$q_{x,*} = \arccos\left(\frac{J_y^2 - (J_x + J_z)^2 - 2J_xJ_z}{2J_xJ_z}\right) = \pi$$

$$q_{y,*} = -\arccos\left(\frac{J_x^2 - (J_y - J_z)^2 + 2J_yJ_z}{2J_yJ_z}\right) = 0$$

Thus $\boldsymbol{q}_*$ and $-\boldsymbol{q}_*$ are the same point because the momentum space live on a torus. The spectrum in this case is drawn in fig. 2.5.

Thus far we have examined only the case in which all the exchange constants are non-zero. Now consider the case in which one of the coupling is zero while the others are non-zero, e.g. $J_z = 0$. In order to be on the

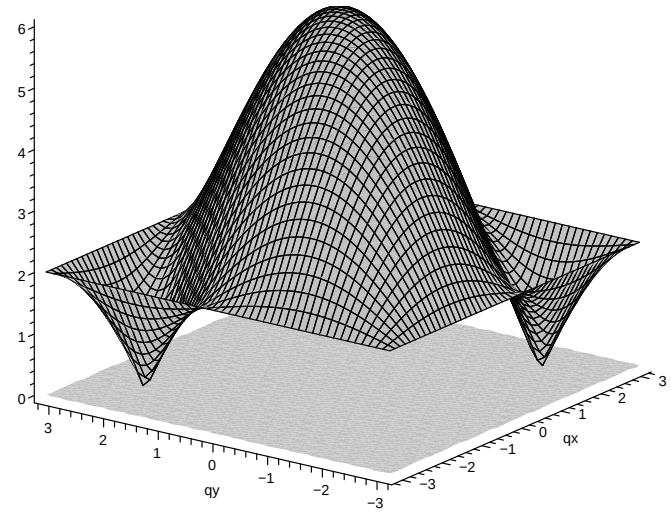


Figure 2.4: The spectrum of the model at the center of gapless phase. The two zero energy points are clearly visible, and in their vicinity the spectrum has a conical singularity.

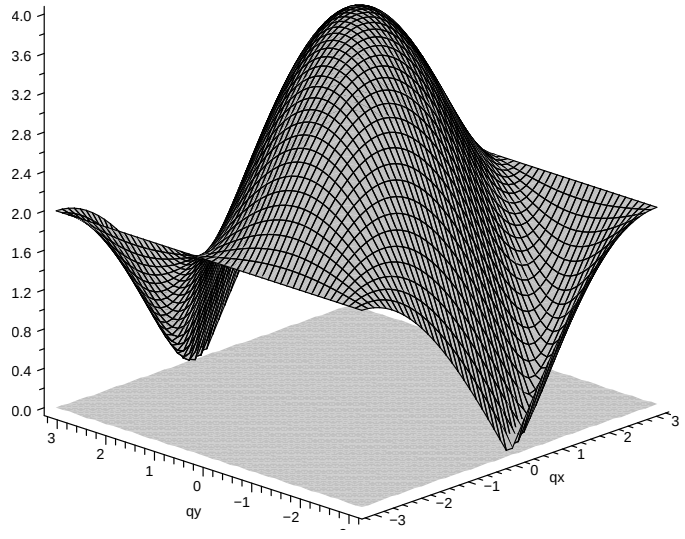


Figure 2.5: The spectrum at the phase boundary $J_x + J_z = J_y$. The two zero solutions are the same one because of momentum is defined on a torus.

phase boundary, it must be $J_x = J_y$, in fact we are on the cross of two phase boundaries, $J_x + J_z = J_y$ and $J_y + J_z = J_x$ on the lowest vertex of the triangular phase B of fig. 2.3.

We cannot use the solutions (2.38), so we take the equations (2.37) and impose $J_z = 0$ and $J_x = J_y$. Obtaining the solutions:

$$\begin{aligned} \sin q_x &= -\sin q_y \\ \cos q_x &= -\cos q_y \end{aligned} \quad \Longrightarrow \quad \begin{aligned} q_x &= q_y + \pi \\ q_x &= -q_y - \pi \end{aligned} \tag{2.44}$$

So there isn't only one point, but a set of points on a straight line, for which the energy is zero. The spectrum is drawn in fig. 2.6.

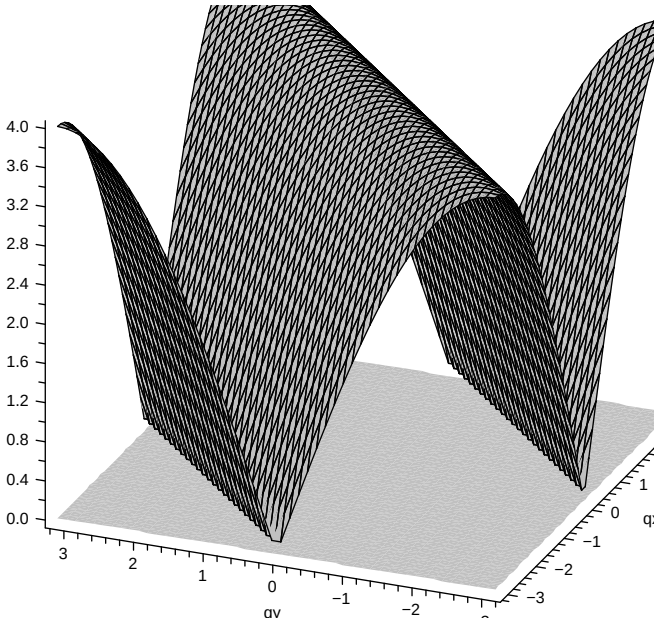


Figure 2.6: The spectrum on the lowest vertex of phase B, i.e. for $J_z = 0$ and $J_x = J_y$. There are a continuum of points for which $\varepsilon(\boldsymbol{q}) = 0$

For more clarity, we derive the spectrum also for another vertex case, the one with $J_x = 0$ and $J_y = J_z$, i.e. the upper right vertex of phase B in fig. 2.3. The equations (2.37) gives:

$$\begin{aligned} \cos q_y &= -1 \\ \sin q_y &= 0 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} q_x &= q_x \\ q_y &= \pm\pi \end{aligned} \tag{2.45}$$

Also in this case there are a continuum of solutions that lie on the boundary of the Brillouin zone as shown in fig.2.7.

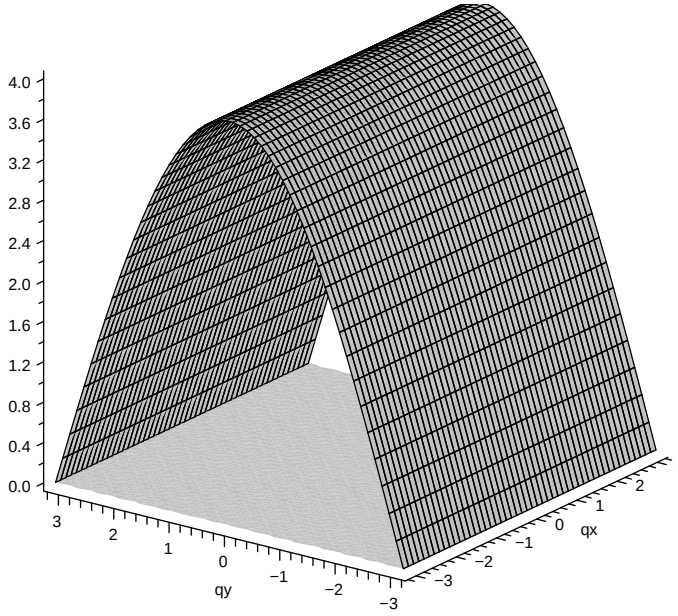


Figure 2.7: The spectrum on the upper right vertex of phase B, $J_x = 0$ and $J_y = J_z$. The two straight line are the same one, indeed they lie on the boundary of Brillouin zone.

2.2.2 Spectrum of the full-vortex state

We now study the other limiting case, the full-vortex configuration, where all the plaquettes has a π-flux or a vortex, i.e. all $w_p = -1$.
To implement that, we can take the $u_{\langle jk\rangle}$ with alternating sign on z-links while the other are all positive, using the same rule as above, i.e. the values of $u_{\langle jk\rangle}$ are those indicated above if $j \in$ white sub-lattice and $k \in$ black one, otherwise are opposite in sign.

We will use the same philosophy of the vortex-free configuration. In this case, in order to have a translational invariant configuration, we have to take as unit cell two z-links for example along the x direction, so that the lattice basis becomes $(2\boldsymbol{n}_x, \boldsymbol{n}_y)$.

With this modifications a site in the original lattice will be identified by this set of index, $j \to (\boldsymbol{r}, l, \lambda)$, where $\boldsymbol{r}$ indicates the position of the new

unit cell, $l = 1, 2$ the z-link in the unit cell and $\lambda = 1, 2$ index the site of the z-link.

In particular we will assign to the z-link variable the value $u = 1$ if $l = 1$, while $u = -1$ if $l = 2$.

The Hamiltonian of free Majorana fermions (2.12) now can be rewritten, considering also the translational invariance:

$$H = \frac{1}{4} \sum_{\boldsymbol{rv}} \sum_{\substack{l,m=1,2 \\ \lambda,\mu=1,2}} A_{l\lambda,m\mu}(\boldsymbol{v}) c_{l\lambda}(\boldsymbol{r}) c_{m\mu}(\boldsymbol{r}+\boldsymbol{v}) \tag{2.46}$$

or in a more compact way:

$$H = \frac{1}{4} \sum_{\boldsymbol{rv}} \sum_{\lambda,\mu=1,2} \boldsymbol{c}_{\lambda}(\boldsymbol{r}) \boldsymbol{A}_{\lambda\mu} \boldsymbol{c}_{\mu}(\boldsymbol{r}+\boldsymbol{v}) \tag{2.47}$$

where the $\boldsymbol{r}$ and $\boldsymbol{v}$ are lattice vectors and we have defined $\boldsymbol{c}_{\lambda} = (c_{1\lambda}, c_{2\lambda})$ and the matrix $\boldsymbol{A}_{\lambda\mu} = [A_{\lambda\mu}]_{lm} = A_{l\lambda,m\mu}$. We can immediately note that $\boldsymbol{A}_{11}(\boldsymbol{v}) = \boldsymbol{A}_{22}(\boldsymbol{v}) = 0$, because exchange between sites of the same color is not allowed.

To solve the Hamiltonian we perform Fourier transform, that need to be redefined because of the new lattice configuration. The Fourier representation of the Majorana operators is:

$$c_{l\lambda}(\boldsymbol{r}) = \frac{1}{\sqrt{C}} \sum_{\boldsymbol{q}} e^{-i\boldsymbol{q}\cdot\boldsymbol{r}} c_{l\lambda}(\boldsymbol{q}) \tag{2.48}$$

where C is the number of new unit cells, q_y take discrete values in the range $[-\pi, \pi]$, while q_x is in the range $[-\pi/2, \pi/2]$ because the x lattice vector is double the y lattice vector. The Majorana operators in momentum representation are:

$$c_{l\lambda}(\boldsymbol{q}) = \frac{1}{\sqrt{C}} \sum_{\boldsymbol{r}} e^{i\boldsymbol{q}\cdot\boldsymbol{r}} c_{l\lambda}(\boldsymbol{r}) \tag{2.49}$$

$$c^{\dagger}_{l\lambda}(\boldsymbol{q}) = \frac{1}{\sqrt{C}} \sum_{\boldsymbol{r}} e^{-i\boldsymbol{q}\cdot\boldsymbol{r}} c_{l\lambda}(\boldsymbol{r}) \tag{2.50}$$

that satisfy $\left\{ c_{l\lambda}(\boldsymbol{p}), c^{\dagger}_{m\mu}(\boldsymbol{q}) \right\} = \delta_{lm} \delta_{\lambda\mu} \delta_{\boldsymbol{pq}}$.

Now we calculate the Fourier representation of the matrix A, defined:

$$[A_{\lambda\mu}]_{lm}(\boldsymbol{q}) = \sum_{\boldsymbol{v}} e^{i\boldsymbol{q}\cdot\boldsymbol{v}} [A_{\lambda\mu}]_{lm}(\boldsymbol{v}) \tag{2.51}$$

and we obtain:

$$\boldsymbol{A}_{11}(\boldsymbol{q}) = \boldsymbol{A}_{22}(\boldsymbol{q}) = 0 \tag{2.52a}$$

as argued above, while the non diagonal terms:

$$\boldsymbol{A}_{12}(\boldsymbol{q}) = \quad 2i \begin{pmatrix} J_z + e^{iq_y} J_y & J_x \\ e^{2iq_x} J_x & -J_z + e^{iq_y} J_y \end{pmatrix} \tag{2.52b}$$

$$\boldsymbol{A}_{21}(\boldsymbol{q}) = -2i \begin{pmatrix} J_z + e^{-iq_y} J_y & e^{-2iq_x} J_x \\ J_x & -J_z + e^{-iq_y} J_y \end{pmatrix} \tag{2.52c}$$

Writing the complete 4x4 matrix $A(\boldsymbol{q})$ we have:

$$\boldsymbol{A}(\boldsymbol{q}) = \begin{pmatrix} 0 & \boldsymbol{A}_{12}(\boldsymbol{q}) \\ \boldsymbol{A}_{21}(\boldsymbol{q}) & 0 \end{pmatrix} \tag{2.53}$$

we note that, as in the vortex-free configuration:

$$\boldsymbol{A}_{21}{}^{\dagger}(\boldsymbol{q}) = A_{12}(\boldsymbol{q}) \tag{2.54}$$

From this property the four eigenvalues of the matrix A, will have specular values, i.e. $\pm\varepsilon_1(\boldsymbol{p})$ and $\pm\varepsilon_2(\boldsymbol{p})$.

In this configuration we obtain four Majorana modes, that can be recombined to form two complex fermions with spectrum $|\varepsilon_1(\boldsymbol{p})|$ and $|\varepsilon_2(\boldsymbol{p})|$ respectively. These are the quasiparticle excitations of the system.

The resulting Hamiltonian after the diagonalization is indeed:

$$H = \sum_{\substack{q_x \in [-\frac{\pi}{2}, \frac{\pi}{2}] \\ q_y \in [-\pi, \pi]}} \sum_{i=1}^{2} |\varepsilon_i(\boldsymbol{q})| \left(b_i^{\dagger}(\boldsymbol{q}) b_i(\boldsymbol{q}) - \frac{1}{2} \right) \tag{2.55}$$

We thus have two level spectrum excitations one of which is always non-zero, while the other can be zero for some value of the exchange constants.

The gapless conditions in this case are:

$$\begin{aligned} J_x^2 + J_y^2 &\geqslant J_z^2 \\ J_y^2 + J_z^2 &\geqslant J_x^2 \\ J_z^2 + J_x^2 &\geqslant J_y^2 \end{aligned} \tag{2.56}$$

The phase diagram is shown in fig. 2.8.

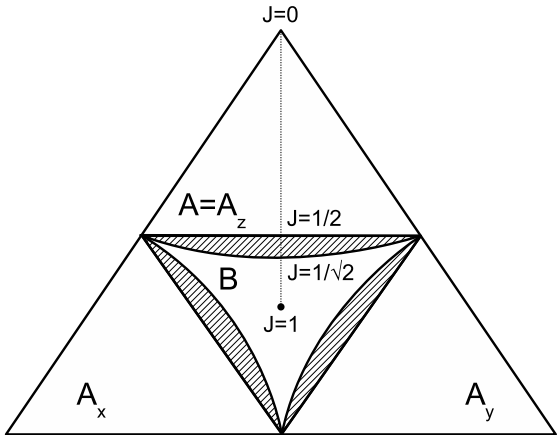


Figure 2.8: The phase diagram for the two configuration studied. The full-vortex one has a restricted gapless phase area. The shaded area indicates the phase boundaries for general vortex configurations.

2.2.3 Dynamical spin correlations and the absence of long range order

Spin correlation are of great importance in magnets, indeed the presence of long range correlations marks a phase transition. We may ask if also in this case, the phase transitiona are driven by local order parameters, such as the magnetization. The answer is no, as we will see below.

In fact the phase transitions in this model are topological [16], that is there is no local order parameter, but it can be characterized in other ways.

We now prove that the correlation between spins are always of short range. We will do this computation in the framework of Majorana fermionization.

If we try to calculate correlations starting from the Hamiltonian of free

Majorana fermions in the background $u_{\langle i,j\rangle}$ field it is not easy to proceed further. We need to go back to the origin, before fixing a specific gauge condition.

We recall the fermionized Hamiltonian of (2.9):

$$H = i \sum_{\langle j,k\rangle} J_{\alpha_{\langle j,k\rangle}} \hat{u}_{\langle j,k\rangle} c_j c_k$$

and we recall also that the extended Hilbert space is decomposed in subspaces, one for each configuration of eigenstates of $\hat{u}_{\langle i,j\rangle}$. So that each eigenstate of the Hamiltonian, is factorized:

$$|\Psi\rangle = |\mathcal{M}_{\mathcal{G}}\rangle \, |\mathcal{G}\rangle \tag{2.57}$$

where $\mathcal{G}$ represent the gauge field sector, i.e. the configuration of u, while $\mathcal{M}_{\mathcal{G}}$ is an eigenstate of the matter sector, i.e. free Majorana fermions in the background $\mathbb{Z}_2$ field.

We now perform a key transformation that was mentioned before, we define complex fermions on the links in this way:

$$f_{\langle j,k\rangle_a} = \frac{1}{2}\left(b^a_j + i b^a_k\right) \tag{2.58a}$$

$$f^\dagger_{\langle j,k\rangle_a} = \frac{1}{2}\left(b^a_j - i b^a_k\right) \tag{2.58b}$$

where we take the convention that j belongs to the white sub-lattice, while k to the black one. These fermions satisfies the usual anticommutations relations:

$$\left\{ f^\dagger_{\langle j,k\rangle}, f_{\langle l,m\rangle} \right\} = \delta_{jl}\delta_{km}$$

The inverse transformations are:

$$b^a_j = f^\dagger_{\langle j,k\rangle_a} + f_{\langle j,k\rangle_a} \qquad \text{for } j \in \text{white sub-lattice} \tag{2.58c}$$

$$b^a_k = i\left(f^\dagger_{\langle j,k\rangle_a} - f_{\langle j,k\rangle_a}\right) \quad \text{for } k \in \text{black sub-lattice} \tag{2.58d}$$

The recombination of Majorana fermions on the link is showed in fig. 2.9. Now we may write the link operators in term of these new fermions:

$$\hat{u}_{\langle j,k\rangle_a} = i b^a_j b^a_k = -\left(f^\dagger_{\langle j,k\rangle_a} + f_{\langle j,k\rangle_a}\right)\left(f^\dagger_{\langle j,k\rangle_a} - f_{\langle j,k\rangle_a}\right) =$$

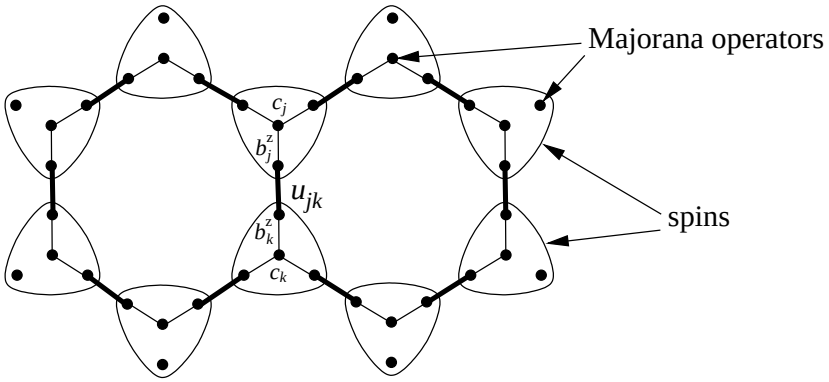


Figure 2.9: The recombination of Majorana fermions on the links.

$$= 2f^\dagger_{\langle jk\rangle_a} f_{\langle jk\rangle_a} - 1 \tag{2.59}$$

Thus all the configurations of u can be chosen to have a definite bond fermion number. We can write therefore:

$$f^\dagger_{\langle jk\rangle_a} f_{\langle jk\rangle_a} |\mathcal{G}\rangle = n_{\langle jk\rangle_a} |\mathcal{G}\rangle \tag{2.60}$$

where $n_{\langle ij\rangle_a}$ is related to the link variable in this way:

$$n_{\langle jk\rangle_a} = \frac{u_{\langle jk\rangle_a} + 1}{2} \tag{2.61}$$

We can now express the spin operators in the extended space, in terms of the new fermions:

$$\sigma^a_j = ib^a_j c_j = i\left(f^\dagger_{\langle jk\rangle_a} + f_{\langle jk\rangle_a}\right) c_j \qquad j \in \text{ white sub-lattice} \tag{2.62a}$$

$$\sigma^a_k = ib^a_k c_k = \left(f_{\langle jk\rangle_a} - f^\dagger_{\langle jk\rangle_a}\right) c_k \qquad k \in \text{ black sub-lattice} \tag{2.62b}$$

The key feature of this transformation is that it connects a single spin site, to three different Majorana operator in three different bonds.

With this representation we can immediately verify what is the effect of σ^a_j when it acts on a state, in the physical states this is a spin flip, but in the extended space it adds a Majorana fermion c_j to the site j and in addition it changes the fermion number on the link $\langle jk\rangle_a$, therefore it changes the sign of the link variable $u_{\langle jk\rangle_a}$. The sign flip of the link variable, can be interpreted

as a creation of two half-quantum vortex in the plaquettes attached to $\langle jk\rangle_a$.

To express this we use this symbolical representation of spin operators:

$$\sigma_j^a = i\hat{\pi}_{p1,\langle jk\rangle_a}\hat{\pi}_{p2,\langle jk\rangle_a}c_j$$
$$\sigma_k^a = \hat{\pi}_{p1,\langle jk\rangle_a}\hat{\pi}_{p2,\langle jk\rangle_a}c_k$$

where we have defined the operators $\hat{\pi}_{p1,\langle jk\rangle}, \hat{\pi}_{p2,\langle jk\rangle}$ that add a π flux, to the plaquette $p1, p2$ attached to the link $\langle jk\rangle_a$. They have the property $\hat{\pi}_p^2 = 1$, because add two π fluxes is equivalent to add zero flux.

We now wish to compute the spin-spin correlation function in physical subspace. From the discussion of section 2.2 we know that spin operators are gauge invariant, so we can compute the spin-spin correlation in any gauge fixed sector and this will be the same as calculating it in the physical gauge invariant sector.

So consider the spin-spin dynamical correlation function in some gauge field configuration $\mathcal{G}$:

$$G_{jk}^{ab}(t) = \langle\mathcal{M}_\mathcal{G}|\langle\mathcal{G}|\sigma_j^a(t)\sigma_k^b(0)|\mathcal{G}\rangle|\mathcal{M}_\mathcal{G}\rangle \tag{2.63}$$

where $\sigma(t) = e^{iHt}\sigma(0)e^{-iHt}$ is the Heisenberg representation of spin operators. We now compute the action of spin operators on the states:

$$\sigma_j^a(0)|\mathcal{G}\rangle|\mathcal{M}_\mathcal{G}\rangle = c_j(0)\left|\mathcal{G}^{ja}\right\rangle|\mathcal{M}_\mathcal{G}\rangle \tag{2.64}$$
$$\sigma_k^b(t)|\mathcal{G}\rangle|\mathcal{M}_\mathcal{G}\rangle = e^{i(H-E)t}c_k(0)\left|\mathcal{G}^{kb}\right\rangle|\mathcal{M}_\mathcal{G}\rangle \tag{2.65}$$

where E is the energy of the eigenstate, given only by the matter sector, and we have defined $\left|\mathcal{G}^{ja}\right\rangle$ as the state with extra π fluxes on the two plaquettes attached to the a-bond starting from the site j.

From this relations we can say that since the vortex are conserved, the correlation function is non-zero only if the added fluxes are on the same plaquettes, i.e. only if $j = k$ and $a = b$ or if j and k are nearest neighbors and $a = b$, i.e.:

$$S_{jk}^{ab}(t) = \begin{cases} g_{jk}(t)\delta_{ab} & jk \text{ nearest neighbors} \\ 0 & \text{otherwise} \end{cases} \tag{2.66}$$

Spin correlation functions between spins are unable to come through nearest

neighbors sites, so they are irresponsible of long range correlations of the phase transitions described in the section above. This is the indicator that it isn't a normal phase transition, because there aren't local order parameters, but it is a topological transition.

Let us give a deeper interpretation of this result. The time evolution of a single spin-flip can be written in this way:

$$e^{iHt}\sigma_j^a(0)e^{-iHt}\left|\Psi\right\rangle = ie^{iHt}c_j(0)e^{-iHt}\hat{\pi}_1\hat{\pi}_2\left|\Psi\right\rangle \tag{2.67}$$

where $|\Psi\rangle$ is a general eigenstate of the Hamiltonian. Since $\hat{\pi}_1$ and $\hat{\pi}_2$ generate two vortex in the plaquettes attached to the link $\langle jk\rangle_a$, the Hamiltonian applied to this state gives the Hamiltonian of free Majorana in the gauge $\mathcal{G}^{ja}$, that we call $H\left[\mathcal{G}^{ja}\right]$ so that, the time evolution is:

$$ie^{iH[\mathcal{G}^{ja}]t}c_j(0)e^{iH[\mathcal{G}^{ja}]}\hat{\pi}_1\hat{\pi}_2\left|\Psi\right\rangle \tag{2.68}$$

This can be re-expressed in terms of evolution in the starting gauge $|\mathcal{G}\rangle$, by considering the extra term in the Hamiltonian, due to the vortex, as an interaction:

$$H\left[\mathcal{G}^{ja}\right] = H\left[\mathcal{G}\right] - 2u_{\langle jk\rangle_a}c_jc_k \tag{2.69}$$

Thus, introducing the interaction representation, we can write:

$$\sigma_j^a(t)\left|\Psi\right\rangle = ic_j(t)T\left(e^{-2u_{\langle jk\rangle_a}J_a\int_0^t d\tau\, c_j(\tau)c_k(\tau)}\right)\hat{\pi}_1\hat{\pi}_2\left|\Psi\right\rangle \tag{2.70}$$

where in the interaction representation the evolution of a generic operator A is:

$$A(t) = e^{iH[\mathcal{G}]t}A(0)e^{-iH[\mathcal{G}]t} \tag{2.71}$$

Thus a spin-flip is a sudden perturbation for the free Majorana fermions and the time ordered term describe how this perturbation evolves the fermion state. After a "long time", related to the time scale of the evolution, the result of this perturbation is a rearrangement of the Majorana vacuum, from the vacuum of the state $|\mathcal{M}_{\mathcal{G}}\rangle$ to that of the state that correspond to the new gauge, with added fluxes, $|\mathcal{M}_{\mathcal{G}^{ja}}\rangle$.

The added Majorana fermion $c_j(t)$, instead, propagates freely as a function of time.

Two spin correlation is the probability to detect the composite (π fluxes

and Majorana fermion) added by a spin flip in site j at time 0 into another site k at time t. Because of π fluxes do not move, this is obviously 0 unless we look at the site linked to j.

To summarize we obtained that independently of the energy of the state, we have an exact confinement of spin flip and an exact deconfinement of Majorana fermions, as they propagate freely.

This phenomenon is known as fractionalization of spin-flip and it happens also in other model, such as in the Heisenberg spin chain, but only as a low energy phenomenon. In the Kitaev model this happens exactly at all energies.

2.3 Fermionization using Jordan-Wigner transformation

In this section we will apply another kind of fermionization to the same model. We will get the spectrum of excitations again and we will verify if we can extract other properties.

We will apply the representation of Jordan-Wigner, this is the only one that doesn't create unphysical states, because it represents each spin with a single fermion.

The problem with this fermionization is that it is not always easy to perform and sometimes it is not convenient, because of phase factors that may emerge, generating a gauge field coupled to fermions.

The feature that let this model suitable for Jordan-Wigner transformations is again in its topology.

Indeed the honeycomb lattice can be deformed to a brick-wall lattice without any change in the topology, as shown in fig. 2.10.

Thus there exists such an order in this 2D model, there are horizontal chains connected by vertical bonds. This let the Jordan-Wigner transformations useful for this model.

Recall the Hamiltonian of Kitaev introduced in § 2.1:

$$H = -J_x \sum_{x\text{-bonds}} \sigma^x_{R_w}\sigma^x_{R_b} - J_y \sum_{y\text{-bonds}} \sigma^y_{R_w}\sigma^y_{R_b} - J_z \sum_{z\text{-bonds}} \sigma^z_{R_w}\sigma^z_{R_b} \qquad (2.72)$$

Now we will denote each site R with the Cartesian coordinate (i, j), where i and j are integer, indeed, if we remove the links, the vertex are disposed

on a square lattice.

Now we can perform the Jordan-Wigner transformation.

Taking advice of the precedent statements we apply an extension of the 1D transformation (1.9), discussed in § 1.2. We follow the contour of fig. 2.11, so we can write for each site:

$$\sigma_{ij}^{\dagger} = 2\left[\prod_{j'<j}\prod_{i'}\sigma_{i'j'}^{z}\right]\left[\prod_{i'<i}\sigma_{i'j}^{z}\right]c_{ij}^{\dagger} \tag{2.73a}$$

$$\sigma_{ij}^{z} = 2c_{ij}^{\dagger}c_{ij} - 1 \tag{2.73b}$$

We now apply this to the Hamiltonian (2.72). For the x-bond terms we have:

$$\begin{aligned}\sigma_{i,j_w}^{x}\sigma_{i+1,j_b}^{x} &= \prod_{i'<i}\sigma_{i',j}^{z}\left(c_{i,j}^{\dagger}+c_{i,j}\right)_w\prod_{i'<i+1}\sigma_{i',j}^{z}\left(c_{i+1,j}^{\dagger}+c_{i+1,j}\right)_b = \\ &= \left(c_{i,j}^{\dagger}+c_{i,j}\right)_w\sigma_{i,j}^{z}\left(c_{i+1,j}^{\dagger}+c_{i+1,j}\right)_b = \\ &= -\left(c_{i,j}^{\dagger}-c_{i,j}\right)_w\left(c_{i+1,j}^{\dagger}+c_{i+1,j}\right)_b\end{aligned}$$

Where we have used:

$$\begin{aligned}\left(c^{\dagger}+c\right)\sigma^{z} = \left(c^{\dagger}+c\right)\left(2c^{\dagger}c-1\right) &= -c^{\dagger}+2cc^{\dagger}c-c = \\ &= -c^{\dagger}-c+2c = -\left(c^{\dagger}-c\right)\end{aligned}$$

Similarly for the y-bond terms we have:

$$\begin{aligned}\sigma_{i,j_b}^{y}\sigma_{i+1,j_w}^{y} &= \prod_{i'<i}\sigma_{i',j}^{z}(-i)\left(c_{i,j}^{\dagger}-c_{i,j}\right)_b\prod_{i'<i+1}\sigma_{i',j}^{z}(-i)\left(c_{i+1,j}^{\dagger}-c_{i+1,j}\right)_w = \\ &= -\left(c_{i,j}^{\dagger}-c_{i,j}\right)_b\sigma_{i,j}^{z}\left(c_{i+1,j}^{\dagger}-c_{i+1,j}\right)_w = \\ &= \left(c_{i,j}^{\dagger}+c_{i,j}\right)_b\left(c_{i+1,j}^{\dagger}-c_{i+1,j}\right)_w\end{aligned}$$

While for the z-bond we have no phase factor, so we can write:

$$\sigma_{i,j_b}^{z}\sigma_{i,j+1_w}^{z} = \left(2c_{i,j}^{\dagger}c_{i,j}-1\right)_b\left(2c_{i,j+1}^{\dagger}c_{i,j+1}-1\right)_w$$

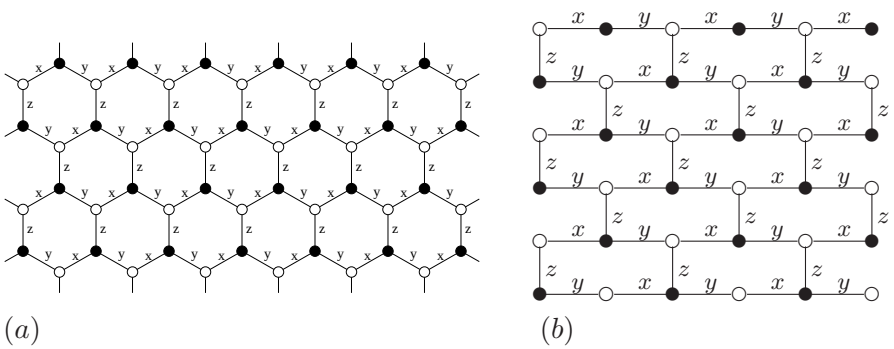


Figure 2.10: A brick wall lattice (b) that is topologically equivalent to the honeycomb (a). From this representation we can see that the lattice is composed by parallel horizontal chains, connected by vertical bonds. Thus a Jordan-Wigner transformation can be performed.

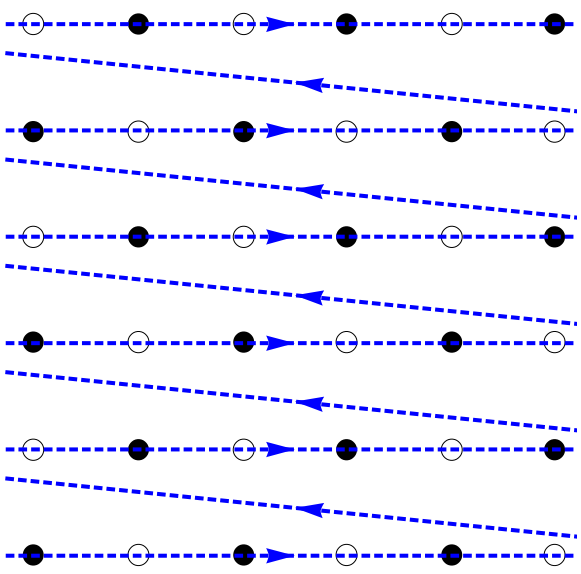


Figure 2.11: The path along which we perform the 1D Jordan-Wigner fermionization.

With these substitutions the Kitaev model (2.72) becomes:

$$\begin{aligned} H = \; & J_x \sum_{x\text{-bonds}} \left(c^\dagger - c\right)_w \left(c^\dagger + c\right)_b \\ & - J_y \sum_{y\text{-bonds}} \left(c^\dagger + c\right)_b \left(c^\dagger - c\right)_w \\ & - J_z \sum_{z\text{-bonds}} \left(2c^\dagger c - 1\right)_b \left(2c^\dagger c - 1\right)_w \end{aligned} \tag{2.74}$$

As expected all the phase factors disappear and the model can be diagonalized with some transformations.

The form of this Hamiltonian suggests the introduction of two Majorana fermions for each site. For the white sites we define:

$$A_w = \frac{\left(c - c^\dagger\right)_w}{i} \qquad B_w = \left(c + c^\dagger\right)_w \tag{2.75a}$$

While for the black sites:

$$A_b = \left(c + c^\dagger\right)_b \qquad B_b = \frac{\left(c - c^\dagger\right)_b}{i} \tag{2.75b}$$

This fermions satisfy the following relations:

$$\begin{gathered} A^2 = B^2 = 1 \\ \left\{A_{i,j}, A_{i',j'}\right\} = \left\{B_{i,j}, B_{i',j'}\right\} = 2\delta_{ii'}\delta_{jj'} \\ \left\{A_{i,j}, B_{i',j'}\right\} = 0 \end{gathered}$$

When substituting into the Hamiltonian (2.74) we have no problem for the x and y bonds, while for the z-bonds we have:

$$\begin{aligned} & \left(2c^\dagger c - 1\right)_b \left(2c^\dagger c - 1\right)_w = \\ & \quad = \left(2\frac{(A - iB)(A + iB)}{4} - 1\right)_b \left(2\frac{(B - iA)(B + iA)}{4} - 1\right)_w = \\ & \qquad\qquad = (iAB)_b (iBA)_w = (iB_b B_w)(iA_b A_w) \end{aligned}$$

After the transformation the model becomes:

$$H = -iJ_x \sum_{x\text{-bonds}} A_w A_b + iJ_y \sum_{y\text{-bonds}} A_b A_w - iJ_z \sum_{z\text{-bonds}} \underbrace{(iB_b B_w)}_{\alpha_r} A_b A_w \tag{2.76}$$

Where we have defined the variable α_r on a z-bond, the vector $\boldsymbol{r}$ marks the position of the bond.

Now we will show that the α_r commute with the Hamiltonian, so we can take them as good quantum numbers.

The demonstration is trivial for the first two term, because a pair of fermions always commute with a pair of different fermions and the same is valid for the third term for all the bonds except the r-th. While for the r-th bond we have:

$$\begin{aligned}[B_b B_w A_b A_w, B_b B_w] &= B_b B_w A_b A_w B_b B_w - B_b B_w B_b B_w A_b A_w = \\ &= B_b B_w \left[A_b A_w, B_b B_w\right] = 0\end{aligned}$$

Thus α_r are good quantum numbers, so once fixed a configuration for them, the Hamiltonian describes again free Majorana fermions in a background of a $\mathbb{Z}_2$ field.

Looking at (2.76) we can see that it is of the same form as (2.12) with all the variable u on x and y bonds fixed to 1.

Thus we remain with only one independent link variable per hexagon, that, as we now see, is related to vortex.

Take the plaquette operator defined in (2.3):

$$I_h = \sigma^y_{1w}\sigma^z_{2b}\sigma^x_{3w}\sigma^y_{4b}\sigma^z_{5w}\sigma^x_{6b} \tag{2.77}$$

where we have added labels that marks the sub-lattice.
As seen before this quantity is conserved in the original kitaev model (2.72).

Let us apply the transformation to the first three terms of (2.77):

$$\begin{aligned}\sigma^y_{1w}\sigma^z_{2b}\sigma^x_{3w} &= \frac{1}{i}\left(c_1^\dagger - c_1\right)_w \sigma^z_{2b}\sigma^z_{1w}\sigma^z_{2b}\left(c_3^\dagger + c_3\right)_w = \\ &= i\left(c_1^\dagger + c_1\right)_w \left(c_3^\dagger + c_3\right)_w = iB_{1w}B_{3w}\end{aligned}$$

where we have used:

$$\left(c_1^\dagger - c_1\right)_w \sigma_{1w}^z = -\left(c_1^\dagger + c_1\right)_w$$

While for the last three terms:

$$\begin{aligned}\sigma_{4b}^y \sigma_{5w}^z \sigma_{6b}^x &= \frac{1}{i}\left(c_4^\dagger - c_4\right)_b \sigma_{6w}^z \sigma_{5w}^z \sigma_{5w}^z \left(c_6^\dagger + c_6\right)_b = \\ &= i\frac{1}{i}\left(c_4^\dagger - c_4\right)_b \frac{1}{i}\left(c_6^\dagger - c_6\right)_b = iB_{4b}B_{6b}\end{aligned}$$

using:

$$\sigma_{6b}^z \left(c_6^\dagger + c_6\right)_b = \left(c_6^\dagger - c_6\right)_b$$

At the end we can write:

$$I_h = (iB_{1w}B_{3w})(iB_{4b}B_{6b}) = (iB_{6b}B_{1w})(iB_{4b}B_{3w}) = \alpha_{61}\alpha_{43} \tag{2.78}$$

This result is consistent with the assertion made before that this fermionization is the same as the precedent Majoranization, but with x and y-link variable fixed to 1.

Indeed the plaquette operator is equal to (2.14) with the substitutions $u = 1$ for x and y links and $u \to \alpha$ on z links.

The difference is that in this case we haven't unphysical states, so the degeneracy and other properties of the model are the correct one.

So we can state that for a given vortex configuration (given by a set of $\{I_h\}$), we have freedom to fix one α for each row in the "brick wall lattice". Once fixed them the conserved quantity $\{I_h\}$ are equivalent to the set $\{\alpha\}$. Thus the ground state degeneracy is 2^{N_r}, where N_r is the number of rows in the brick-wall lattice.

The degeneracy is different if the system is on a torus, indeed in that case the degeneracy is lifted [29].

In order to diagonalize the Hamiltonian (2.76) we introduce a fermion in each z-bonds, that we will identify by the vector position $\boldsymbol{r}$:

$$d_r = \frac{1}{2}(A_w + iA_b) \qquad d_r^\dagger = \frac{1}{2}(A_w - iA_b) \tag{2.79}$$

where A_w is the Majorana fermion on the white site of the z-link, while A_b the one on the black site.

Denoting by $\boldsymbol{n}_x$ the unit vector that connects two z-bonds crossing a x-bond and by $\boldsymbol{n}_y$ the unit vector that connects two z-bonds crossing a y-bond, as shown in fig. 2.12. We can write the Hamiltonian (2.76) in this way:

$$\begin{aligned} H = \; & J_x \sum_r \left(d_r^\dagger + d_r\right)\left(d_{r+\boldsymbol{n}_x}^\dagger - d_{r+\boldsymbol{n}_x}\right) + \\ & + J_y \sum_r \left(d_r^\dagger + d_r\right)\left(d_{r+\boldsymbol{n}_y}^\dagger - d_{r+\boldsymbol{n}_y}\right) + \\ & + J_z \sum_r \alpha_r \left(2 d_r^\dagger d_r - 1\right) \end{aligned}$$

This is a hopping model Hamiltonian with site dependent chemical potential.

As stated above the ground state of this model is the state without vortex, this is with all $I_h = +1$, which correspond to various set of $\{\alpha\}$, one for each choice of one α in a row.

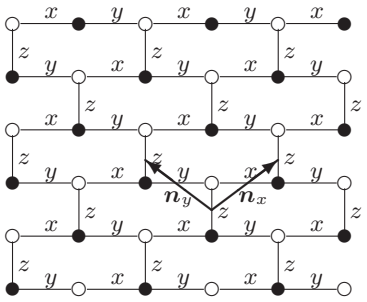


Figure 2.12: The unit vectors that connect two z-bonds. They are equivalent to those that was defined in the Majoranization of §2.2

2.3.1 Spectrum of the vortex-free state and emergence of BCS Hamiltonian

To find the spectrum we can choose all the $\alpha = +1$, hence the model is translational invariant and we can resolve by Fourier transformation:

$$\begin{aligned} d_r &= \frac{1}{\sqrt{\Omega}} \sum_q e^{i\boldsymbol{q}\boldsymbol{r}} d_q \\ d_r^\dagger &= \frac{1}{\sqrt{\Omega}} \sum_q e^{-i\boldsymbol{q}\boldsymbol{r}} d_q^\dagger = \frac{1}{\sqrt{\Omega}} \sum_q e^{i\boldsymbol{q}\boldsymbol{r}} d_{-q}^\dagger \end{aligned}$$

For the x terms we have

$$\sum_r \left(d_r^\dagger + d_r\right)\left(d_{r+\boldsymbol{n}_x}^\dagger - d_{r+\boldsymbol{n}_x}\right) =$$
$$= \frac{1}{\Omega}\sum_{q,q'} \underbrace{\sum_r e^{i(\boldsymbol{q}+\boldsymbol{q'})\boldsymbol{r}}}_{\Omega\delta_{q,-q'}} e^{i\boldsymbol{q}\boldsymbol{n}_x}\left(d_{-q}^\dagger + d_q\right)\left(d_{-q'}^\dagger - d_{q'}\right) =$$
$$= \sum_q \left[-2\cos q_x d_q^\dagger d_q + \frac{1}{2}2i\sin q_x\left(d_q^\dagger d_{-q}^\dagger + \text{h.c.}\right)\right]$$

After computing the Fourier transform we obtain the following Hamiltonian for the ground state

$$H_g = \sum_q \left[\varepsilon_q d_q^\dagger d_q + i\frac{\Delta_q}{2}\left(d_q^\dagger d_{-q}^\dagger + \text{h.c.}\right)\right]$$

where:

$$\varepsilon_q = 2J_z - 2J_x\cos q_x - 2J_y\cos q_y$$
$$\Delta_q = 2J_x\sin q_x + 2J_y\sin q_y$$

This has the form of a BCS model and after computing a Bogoliubov transformation we obtain the quasiparticle excitations, (for the details of calculation see Appendix A):

$$E_q = \sqrt{\varepsilon_q^2 + \Delta_q^2} \tag{2.80}$$

Remarkably this is the Hamiltonian of a p-wave superconductor with BCS pairing.
This is a very important result because also in p-wave superconductors there are possible vortex excitations as in the Kitaev model.

Vortex in p-wave superconductors obey fractional statistics, in particular the are non-abelian "anyons" [20].

As showed by A. Kitaev [15] also in the honeycomb models there are abelian and non-abelian anyons, corresponding respectively to vortex excitations in the A_i phase and in B phase with a magnetic field.

The mapping to a BCS theory is yet to be better understanded and we think that it can be an interesting line of research to understand the nature and possible use of these anyons.

For example, an interesting feature of p-wave superconductors is that in the core of each vortex there is an unpaired Majorana mode [30, 21, 23]. What happens in the Kitaev model?

We now go ahead to find if the spectrum has a gap and in what condition. The spectrum is gapless, if $\varepsilon(\boldsymbol{q}) = 0$ and $\Delta(\boldsymbol{q}) = 0$.
These conditions are the same as (2.37) considered in the section of Majoranization, if we can change the sign of J_z, but, changing the sign of J_z, it is equivalent to change all the sign of α_r and this transformation doesn't modify the vortex configuration. Thus we have the same phase diagram as in fig. 2.3.

As expected all the information on the spectrum are the same in the two models.

2.3.2 Ground state in real space

From this representation we can also get directly the exact physical ground state. From Appendix A we can get the ground state in term of the fermions d:

$$|g\rangle = \prod_k{}' \left(u_k + v_k d^\dagger_{-k} d^\dagger_k\right) |0\rangle \tag{2.81}$$

where the prime means that the product is performed such that pair $k, -k$ must be considered only once.

We can get the ground space in terms of spin operators by reverting all the transformations made before and expressing the vacuum of fermions in term of a reference spin state $|\phi\rangle$, for example the one with all spin up in the σ_z basis, i.e. $|\phi\rangle = |\uparrow\uparrow \cdots \uparrow\rangle$.

The explicit computation is very long and is done by H. Chen and Z. Nussinov [29].

Chapter 3

Conclusions, remarks and further outlook

In this work we have shown how the formalism of fermionization permits to solve exactly some models.

It is interesting to explore if this methods are generalizable for other spin systems.
Majorana representation is easier to apply, but it is not always convenient, because of it needs an extended Hilbert space.

The Jordan-Wigner representation is useful because it needs no constraints, mapping each spin to a single fermion. But it is not always applicable. Consider for example a star lattice, i.e. four chain intersecting in one site, for this it is impossible to apply the Jordan-Wigner transformation used for the Kitaev model.
The problem is in the ordering. For dimensions greater than 1 an explicit ordering does not exist, but if we can find a path that covers all the sites, without auto-intersections, as we have done for the honeycomb, then we may perform the Jordan-Wigner transformation along that path, although it is not always convenient.

We have examined in detail the Kitaev model on a honeycomb lattice. This is a two dimensional model that exhibits an important topological behavior and we have shown that it is exactly soluble by fermionization.
The remarkable result we obtained is that we obtain the same spectrum either if we use the Majorana representation of spins, proposed by A. Tsvelik [9] and A. Kitaev [15], or the Jordan Wigner representation.

By analysis of the spectrum we have pointed the emergence of Quantum Phase Transitions, that are not related to any local order parameter, so that it is not applicable the Landau theory of phase transitions, thaw was applicable, for example, in the Ising model. This kind of order is called topological and the characterization of it is object of intensive study in condensed matter theory [16, 17, 31, 27, 32].
As exposed in the works of Feng [32] and Chen [27], Majorana fermions can be used to define fermions on bonds, the dual lattice. This sometimes permits the definition of local order parameters and the Landau theory may be applied. These order parameters (local in the dual lattice) correspond to topological string order parameters (non-local) in the real lattice.

Although they give the same escitation spectrum, Majorana and Jordan-Wigner representation are not the same thing, because the former introduce unphysical states that do not give the correct states. The latter, instead, do not require constraints, so it may be used to obtain exact states [29].

Using the Jordan-Wigner representation we have also shown the emergence of a p-wave BCS theory from the Kitaev model.

The key feature of the model, beside being exactly solvable, is that it contains a background $\mathbb{Z}_2$ field interacting with Majorana fermions on sites. This field is related to vortex.

For each vortex configuration we can obtain the spectrum of the fermions, using the method used above [33]. We can get the vortex excitation spectrum by taking the ground state energy for each configuration.

Vortices are topological excitations and has been shown that a spin-flip on a site may create a pair of vortices.

In p-wave superconductors vortex are topological excitations. They are related to zero-mode Majorana fermions localized in the core of vortices.
Since two Majorana zero-mode in two different vortices form a single fermionic state, there is an entanglement between spatially separated vortices. This entanglement is the source of a non-trivial mutual statistics when moving a vortex around another, a non-Abelian anyonic statistics [20, 30].

Emerging Anyons are also present in the Kitaev model, in particular they are the vortex excitations and can be Abelian or non-Abelian. To study Anyons excitation spectrum it is necessary that fermion spectrum is gapped. In the model studied this happens only in the A_i phases, and it can be shown that these vortex are Abelian.

Introducing a time reversal breaking term in the Hamiltonian, e.g. interactions between three spins, opens a gap also in the B phase [15, 34], and the Anyons in this case are non-abelian.

Anyons are of particular importance for Topological Quantum Computation, as they permits fault-tolerant quantum computation [18], that is they are insensitive to local perturbations due to their topological character, but, to permit all the logical gates, they have to be non-abelian one [15]. This is the reason why this model is important.

A major characterization of the zero-mode Majorana bound to the vertices is necessary. In particular it is important to understand how unpaired Majorana fermions in the vortex are entangled and if they are connected to unpaired Majorana zero-modes on the boundaries [26, 35, 36].

Appendix A

Solution of a BCS Hamiltonian

Consider a quadratic Hamiltonian of the BCS form:

$$H = \sum_q \left[\varepsilon_q d_q^\dagger d_q + \frac{1}{2} \left(i\Delta_q d_q^\dagger d_{-q}^\dagger - i\Delta_q d_{-q} d_q \right) \right] \tag{A1}$$

where d_q are fermion operators, ε_q is the energy for a particle with momentum q and Δ_q is a real function of the momentum that describe a BCS pairing.

To have a physical meaning ε_q must be symmetric, i.e. $\varepsilon_{-q} = \varepsilon_q$, while we consider that Δ_q is antisymmetric, $\Delta_{-q} = -\Delta_q$.

The phase i can be removed by a global gauge transformation:

$$d_q^\dagger \longrightarrow e^{-i\frac{\pi}{4}} d_q^\dagger \qquad\qquad d_q \longrightarrow e^{i\frac{\pi}{4}} d_q$$

so that (A1) becomes:

$$H = \sum_q \left[\varepsilon_q d_q^\dagger d_q + \frac{\Delta_q}{2} \left(d_q^\dagger d_{-q}^\dagger + d_{-q} d_q \right) \right] \tag{A2}$$

This can be diagonalized using Bogoliubov transformations, but before it is convenient to write it in a more symmetric way, using the property of ε_q:

$$H = \sum_q \left[\frac{\varepsilon_q}{2} \left(d_q^\dagger d_q + d_{-q}^\dagger d_{-q} \right) + \frac{\Delta_q}{2} \left(d_q^\dagger d_{-q}^\dagger + d_{-q} d_q \right) \right]$$

so that it is of the form:

$$H = \sum_q \left[\varepsilon_0 \left(a_q^\dagger a_q + b_q^\dagger b_q\right) + \varepsilon_1 \left(a_q^\dagger b_q^\dagger + b_q a_q\right)\right]$$

where $a_q = d_q$ and $b_q = d_{-q}$.

Now we perform the following Bogoliubov transformations, defining these new fermionic operators:

$$A_k = u_k a_k + v_k b_k^\dagger \qquad A_k^\dagger = u_k a_k^\dagger + v_k b_k \tag{A3a}$$
$$B_k = v_k a_k^\dagger - u_k b_k \qquad B_k^\dagger = v_k a_k - u_k b_k^\dagger \tag{A3b}$$

where u_k and v_k are coefficient that can be taken to be real and they need to be solved in order to get a diagonal Hamiltonian. These coefficient must also satisfy the condition

$$u_k^2 + v_k^2 = 1 \tag{A4}$$

to give the right anticommutators of A_k and B_k, indeed:

$$\left\{A_k, A_k^\dagger\right\} = \left\{B_k, B_k^\dagger\right\} = u_k^2 + v_k^2 = 1.$$

While other anticommutators are automatically satisfied by definitions (A3).

Using the condition (A4) we can write the inverse transformations:

$$a_k = u_k A_k + v_k B_k^\dagger \qquad a_k^\dagger = u_k A_k^\dagger + v_k B_k \tag{A5a}$$
$$b_k = v_k A_k^\dagger - u_k B_k \qquad b_k^\dagger = v_k A_k + u_k B_k^\dagger \tag{A5b}$$

Now we substitute in the Hamiltonian obtaining:

$$H = \sum_k \left\{ \left[\frac{\varepsilon_k}{2}\left(u_k^2 - v_k^2\right) + \Delta_k u_k v_k\right] \left(A_k^\dagger A_k - B_k B_k^\dagger\right) + \right.$$
$$\left. + \left[\varepsilon_k u_k v_k - \frac{\Delta_k}{2}\left(u_k^2 - v_k^2\right)\right] \left(A_k^\dagger B_k^\dagger + B_k A_k\right)\right\} \tag{A6}$$

Now we impose that the non diagonal term vanishes:

$$\varepsilon_k u_k v_k - \frac{\Delta_k}{2}\left(u_k^2 - v_k^2\right) = 0 \tag{A7}$$

From the condition (A4) we can write

$$u_k = \cos\theta_k, \quad v_k = \sin\theta_k$$

where θ_k is a parameter to be determined. With these substitutions the equation (A7) becomes

$$\varepsilon_k \sin 2\theta_k - \Delta_k \cos 2\theta_k = 0$$

that gives

$$\tan 2\theta_k = \frac{\Delta_k}{\varepsilon_k} \tag{A8}$$

In order to get the diagonal part of the Hamiltonian, we solve for $u_k^2 - v_k^2$ and for $u_k v_k$, finding

$$u_k^2 - v_k^2 = \cos 2\theta_k = \frac{1}{\sqrt{1+\tan^2 2\theta_k}} = \frac{\varepsilon_k}{E_k} \tag{A9a}$$

$$u_k v_k = \frac{1}{2}\sin 2\theta_k = \frac{1}{2}\tan 2\theta_k \cos 2\theta_k = \frac{1}{2}\frac{\Delta_k}{E_k} \tag{A9b}$$

where $E_k = \sqrt{\varepsilon_k^2 + \Delta_k^2}$ and has been taken positive value for square root so that $A^\dagger$ and $B^\dagger$ create states with positive energy, i.e. excitations. From the first of (A9) and using condition (A4) we can write for the constants u_k, v_k:

$$u_k^2 = \frac{1}{2}\left(1 + \frac{\varepsilon_k}{E_k}\right) \tag{A10a}$$

$$v_k^2 = \frac{1}{2}\left(1 - \frac{\varepsilon_k}{E_k}\right) \tag{A10b}$$

and from the second we can get some additional properties, indeed if we compute $u_{-k}v_{-k}$, we get, using the symmetries, $E_{-k} = E_k$ and $\Delta_{-k} = -\Delta_k$:

$$u_{-k}v_{-k} = -\frac{\Delta_k}{E_k} \tag{A11}$$

Thus one of the two constants must change his sign when $k \to -k$, while the other remains unchanged, we will take:

$$u_{-k} = -u_k\,, \qquad v_{-k} = v_k \tag{A12}$$

With this choose, we can immediately see from transformations (A3) that:

$$B_{-k}^{\dagger} = A_k^{\dagger} \qquad B_{-k} = A_k \tag{A13}$$

Using these relations we can compute the diagonal part of the Hamiltonian (A6):

$$\frac{\varepsilon_k}{2}\left(u_k^2 - v_k^2\right) + \Delta_k u_k v_k = \frac{1}{2}\frac{\varepsilon_k^2 + \Delta_k^2}{E_k} = \frac{E_k}{2} \tag{A14}$$

so that the Hamiltonian becomes, neglecting constant terms:

$$H = \sum_k \frac{E_k}{2}\left(A_k^{\dagger} A_k - B_k B_k^{\dagger}\right) \tag{A15}$$

and using the relations (A13) becomes:

$$H = \sum_k E_k \left(A_k^{\dagger} A_k - \frac{1}{2}\right) \tag{A16}$$

The excitations are expressed in terms of quasiparticle $A^{\dagger}$.

The ground state of that Hamiltonian is the one without any quasiparticle, so when acting with annihilating operators gives zero:

$$A_k \left|g\right\rangle = 0\,, \quad \forall k \tag{A17}$$

It is now interesting to express the ground state in terms of the original fermions. It can be expressed in this way:

$$\left|g\right\rangle = \prod_k A_k \left|0\right\rangle = \prod_k{}' A_k A_{-k} \left|0\right\rangle \tag{A18}$$

where $\left|0\right\rangle$ is the vacuum of the original fermions, i.e. $a_k \left|0\right\rangle = b_k \left|0\right\rangle = 0$ for each k, and the prime over the product indicate that every pair $k, -k$ is to be considered once. This is effectively the ground state because application of annihilating operator to $\left|g\right\rangle$ gives indeed zero.

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$$\left|g\right\rangle = \prod_k A_k \left|0\right\rangle = \prod_k{}' A_k A_{-k} \left|0\right\rangle \tag{A19}$$

where $|0\rangle$ is the vacuum of the original fermions, i.e. $a_k |0\rangle = b_k |0\rangle = 0$ for each k, and the prime over the product indicate that every pair $k, -k$ is to be considered once. This is effectively the ground state because application of annihilating operator to $|g\rangle$ gives indeed zero.

So we can express the ground state of the Hamiltonian (A1) in term of the vacuum of original fermions, by computing:

$$\begin{aligned} A_k A_{-k} |0\rangle &= \left(u_k d_k + v_k d_{-k}^\dagger\right)\left(v_k d_k^\dagger - u_k d_{-k}\right) = \\ &= \left(u_k v_k + v_k^2 d_{-k}^\dagger d_k^\dagger\right) \end{aligned} \tag{A20}$$

Then imposing the normalization $\langle g \,|\, g\rangle \equiv 1$, we have, for each factor:

$$\begin{aligned} {}_k\langle g \,|\, g\rangle_k &= C^2 v_k^2 \langle 0| \left(u_k + v_k d_k d_{-k}\right)\left(u_k + v_k d_{-k}^\dagger d_k^\dagger\right) = 1 \\ &= C^2 v_k^2 (u_k^2 + v_k^2) = 1 \quad \Longrightarrow \quad C = \frac{1}{v_k} \end{aligned}$$

Obtaining:

$$|g\rangle = \prod_k{}' \left(u_k + v_k d_{-k}^\dagger d_k^\dagger\right) |0\rangle \tag{A21}$$

If, for some k, E_k take the value 0, then the spectrum is gapless and there are long range correlations, while if it is not the spectrum has a gap and excitations have minimum energy.

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